

HELICES ON DEL PEZZO SURFACES AND TILTING CALABI-YAU ALGEBRAS

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ABSTRACT. We study tilting for a class of Calabi-Yau algebras associated to helices on Fano varieties. We do this by relating the tilting operation to mutations of exceptional collections. For helices on del Pezzo surfaces the algebras are of dimension three, and using an argument of Herzog, together with results of Kuleshov and Orlov, we obtain a complete description of the tilting process in terms of quiver mutations.

1. INTRODUCTION

In the context of homological algebra, tilting is a fundamental construction that relates neighbouring t-structures in a triangulated category. It first appears at this level of generality in a paper of Happel, Reiten and Smalø [12]. The basic idea however goes back much further, and the name was introduced by Brenner and Butler [1], who studied the process in the context of categories of representations of quivers. More recently tilting for three-dimensional Calabi-Yau algebras has been related to cluster mutations [19, 20, 22], and to Seiberg duality in theoretical physics [2].

The aim of this paper is to study tilting for a class of Calabi-Yau algebras associated to helices of coherent sheaves on Fano varieties. Our basic tool will be the theory of exceptional collections, developed in the Rudakov seminar [26]. Before stating our main result (Theorem 1.7) we will recall the definition of tilting in the context of algebras defined by quivers with relations, and introduce the class of algebras to be studied.

Notation. All algebras and varieties will be over the complex numbers. All modules will be right modules. We write $D(X)$ for the bounded derived category of coherent sheaves on a variety X , and $D(A)$ for the homotopy category of bounded complexes of finitely-generated projective modules over an algebra A . If A is noetherian and of finite global dimension this is equivalent to the bounded derived category of finitely-generated A -modules. Finally, we write $D_{\text{fin}}(A) \subset D(A)$ for the subcategory of complexes with finite-dimensional cohomology.

1.1. Quivers and tilting. Let Q be a quiver specified in the usual way by a set of vertices Q_0 , a set of arrows Q_1 , and source and target maps $s, t: Q_1 \rightarrow Q_0$. We shall always assume that the number of arrows n_{ij} from vertex i to vertex j is finite. The path algebra $\mathbb{C}Q = \bigoplus_{l \geq 0} \mathbb{C}Q_l$ is graded by path length, and the degree zero part $\mathbb{S} = \mathbb{C}Q_0$ is a semisimple ring with a basis of orthogonal idempotents e_i indexed by the vertices Q_0 .

Given a two-sided ideal $I \triangleleft \mathbb{C}Q$ generated by linear combinations of paths of length at least two, let

$$A = A(Q, I) = \mathbb{C}Q/I$$

denote the corresponding quotient algebra. Then A is an augmented \mathbb{S} -algebra with augmentation ideal $A_+ \triangleleft A$ spanned by paths of positive length. The underlying quiver Q is determined by the augmented algebra A , since

$$(1) \quad n_{ij} = \dim_{\mathbb{C}} e_j \cdot (A_+/(A_+)^2) \cdot e_i.$$

We shall refer to augmented algebras of this form as *quiver algebras*.

Suppose that $A = A(Q, I)$ is a quiver algebra. To each vertex $i \in Q_0$ there corresponds an indecomposable projective module $P_i = e_i A$, and a one-dimensional simple module S_i , on which all elements of A_+ act by zero. It is easy to see that

$$(2) \quad n_{ji} = \dim_{\mathbb{C}} \operatorname{Ext}_A^1(S_i, S_j).$$

There is also a canonical map

$$\bigoplus_{j \in Q_0} P_j^{\oplus n_{ji}} \longrightarrow P_i$$

which, viewed as a complex of modules concentrated in degree 0 and 1, defines an object R_i of the category $K^b \operatorname{Proj}(A)$.

Definition 1.1. We say that quiver algebras $A = A(Q, I)$ and $A' = A(Q', I')$ are related by a *vertex tilt* if there is an equivalence of categories

$$\Psi: D(A) \rightarrow D(A'),$$

a bijection $\psi: Q_0 \rightarrow Q'_0$, and a vertex $i \in Q_0$ such that

$$\Psi(P_j) = P'_{\psi(j)} \text{ for } j \neq i \quad \text{and} \quad \Psi(R_i) = P'_{\psi(i)}.$$

More precisely we say that A' is the left tilt of A at the vertex i , and that A is the right tilt of A' at the vertex $\psi(i)$.

Remark 1.2. Suppose that the quivers underlying the algebras A and A' have no loops. The equivalence Ψ restricts to an equivalence

$$\Psi: D_{\text{fin}}(A) \rightarrow D_{\text{fin}}(A').$$

Define objects $U_j \in D_{\text{fin}}(A)$ for $j \in Q_0$ by the relation $\Psi(U_j) = S_{\psi(j)}$. Then $U_i = S_i[-1]$, whereas for $j \neq i$ the object U_j is the universal extension

$$(3) \quad 0 \longrightarrow S_j \longrightarrow U_j \longrightarrow \operatorname{Ext}_A^1(S_i, S_j) \otimes S_i \longrightarrow 0.$$

For a proof of these facts see Lemma A.4. It is an easy consequence of this that when the algebras A and A' are noetherian of finite global dimension the inverse image under Ψ of the standard t-structure on $D_{\text{fin}}(A')$ is related to the standard t-structure on $D_{\text{fin}}(A)$ by an abstract tilt in the sense of [12]. The relevant torsion pair has torsion part consisting of direct sums of the module S_i .

1.2. Tilting CY_3 quiver algebras. We shall say that a quiver algebra $A = A(Q, I)$ is (weakly) Calabi-Yau of dimension d , or just CY_d , if A has global dimension d and the shift functor $[d]$ is a Serre functor on $D_{\text{fin}}(A)$. This last condition means that there are functorial isomorphisms

$$\operatorname{Hom}_A(M, N) \cong \operatorname{Hom}_A(N, M[d])^*$$

for all objects $M, N \in D_{\text{fin}}(A)$. We shall be particularly interested in the case of CY_3 quiver algebras. The combinatorics of the tilting process for such algebras can be described by a rule known as quiver mutation, as we now explain.

The CY_3 condition implies that the Euler matrix

$$\chi(i, j) = \chi(S_i, S_j) = \sum_{i=0}^3 (-1)^i \dim_{\mathbb{C}} \operatorname{Ext}_A^i(S_i, S_j).$$

is skew-symmetric. Suppose that the underlying quiver Q has no loops or oriented 2-cycles. Then by (2)

$$n_{ij} = \begin{cases} \chi(i, j) & \text{if } \chi(i, j) > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Thus the structure of the quiver is completely determined by the Euler matrix.

Lemma 1.3. *Suppose $A = A(Q, I)$ and $A' = A(Q', I')$ are CY_3 quiver algebras related by a tilt at the vertex $i \in Q_0$ as in Definition 1.1. Then the Euler form for the algebra A' is*

$$\chi(\psi(j), \psi(k)) = \begin{cases} -\chi(j, k) & \text{if } i \in \{j, k\}, \\ \chi(j, k) & \text{if } i \notin \{j, k\} \text{ and } \chi(i, j) \chi(i, k) \geq 0, \\ \chi(j, k) + |\chi(i, j)| \cdot \chi(i, k) & \text{if } i \notin \{j, k\} \text{ and } \chi(i, j) \chi(i, k) \leq 0, \end{cases}$$

Proof. This follows easily from (3) using the additivity property of χ on exact sequences. \square

Thus if we also assume that Q' has no loops or 2-cycles the quiver Q' is completely determined by the quiver Q and the vertex i . The resulting transformation law $\mu_i: Q \mapsto Q'$ is called *quiver mutation* and is easily checked to be an involution. It is very important to note however that one does not usually know *a priori* that the quiver Q' has no 2-cycles, so that without further information, one cannot guarantee that the quiver underlying the algebra A' is indeed given by the above rule.

Remark 1.4. Keller and Yang [20] have recently gone further and described how the ideals of relations I and I' are related under a tilt. Take assumptions as in Lemma 1.3. For simplicity let us also assume that A and A' are graded. Work of Bocklandt [3, Theorem 3.1] then shows that the relations can be encoded in compact form in a potential. Thus we can write

$$A = A(Q, W) = \mathbb{C}Q/(\partial_a W : a \in Q_1)$$

for some non-uniquely defined element $W \in \mathbb{C}Q/[\mathbb{C}Q, \mathbb{C}Q]$. We can similarly write $A' = A(Q', W')$. Then Keller and Yang [20, Theorem 3.2] (see also [19, Theorem 9.2]) show that the potentials W and W' can be chosen so that (Q', W') is obtained from (Q, W) by a simple combinatorial rule described explicitly by Derksen, Weyman and Zelevinsky [9].

1.3. Rolled-up helix algebras. Let Z be a smooth projective variety and let ω_Z denote its canonical line bundle. In the cases of most interest to us Z will be a Fano variety, which is to say that the dual of ω_Z is ample.

Definition 1.5. A *helix of sheaves* on Z of period n is an infinite collection of coherent sheaves $\mathbb{H} = (E_i)_{i \in \mathbb{Z}}$ such that for all $i \in \mathbb{Z}$ one has

- (i) $(E_{i+1}, \dots, E_{i+n})$ is a full exceptional collection,
- (ii) $E_{i-n} = E_i \otimes \omega_Z$.

We recall many aspects of the theory of exceptional collections and helices in Sections 2 and 3 below. Associated to a helix \mathbb{H} is a graded algebra

$$A(\mathbb{H}) = \bigoplus_{k \geq 0} \prod_{j-i=k} \text{Hom}_Z(E_i, E_j)$$

equipped with a \mathbb{Z} -action induced by twisting by ω_Z

$$- \otimes \omega_Z: \text{Hom}_Z(E_i, E_j) \rightarrow \text{Hom}_Z(E_{i-n}, E_{j-n}).$$

The rolled-up helix algebra $B(\mathbb{H})$ is defined to be the subalgebra of $A(\mathbb{H})$ consisting of elements invariant under this action.

A helix $\mathbb{H} = (E_i)_{i \in \mathbb{Z}}$ is called *geometric* if for all $i < j$ one has the additional condition

$$\mathrm{Hom}_Z^k(E_i, E_j) = 0 \text{ unless } k = 0.$$

The following result was mostly proved in [8, Prop. 4.1], see also [27, Prop. 7.2]. For completeness we give a proof in Section 3.3.

Theorem 1.6. *Let $B = B(\mathbb{H})$ be the rolled-up helix algebra of a geometric helix on a smooth projective Fano variety Z of dimension $d - 1$. Then B is a graded CY_d quiver algebra which is noetherian and finite over its centre. Moreover, there are equivalences of categories $D(B) \cong D(Y)$ where Y is the total space of the line bundle ω_Z .*

The tilting operation for rolled-up helix algebras was studied in [8] under the additional assumption

$$(4) \quad \mathrm{rank} K(Z) = \dim Z + 1.$$

This case is much simpler than the general one. In particular, the quiver underlying $B(\mathbb{H})$ is always an oriented n -gon with various numbers of arrows along the edges, and the combinatorics of the tilting process can be described explicitly using an action of the affine braid group.

In this paper we shall be most interested in the case when Z is a del Pezzo surface, that is a Fano variety of dimension 2. Recall that any such surface is isomorphic either to

- (a) $\mathbb{P}^1 \times \mathbb{P}^1$, or
- (b) the blow-up of \mathbb{P}^2 at $m \leq 8$ points.

The only case satisfying (4) is $Z = \mathbb{P}^2$.

Thanks to work of Karpov and Nogin [17] it is known that geometric helices exist on all del Pezzo surfaces (see Example 8.6 below). The main content of the following result is that it provides an interesting class of CY_3 algebras, closed under vertex tilts, where the quiver mutation rules really apply, in that one knows *a priori* that there are no loops or 2-cycles.

Theorem 1.7. *Let $B = B(\mathbb{H})$ be the rolled-up helix algebra of a geometric helix \mathbb{H} on a del Pezzo surface Z . Then B is a graded CY_3 quiver algebra whose underlying quiver Q has no loops or oriented 2-cycles. For any vertex i of Q there is another geometric helix \mathbb{H}' on Z such that the algebra $B(\mathbb{H}')$ is the (left and right) tilt of $B(\mathbb{H})$ at the vertex i .*

It follows from Theorem 1.7 and Remark 1.4 that tilting for rolled-up helix algebras on del Pezzo surfaces is completely described by the combinatorial quiver mutation process of [9]. It is worth noting that the same combinatorics was considered earlier by physicists studying Seiberg duality for quiver gauge theories, see for example [10].

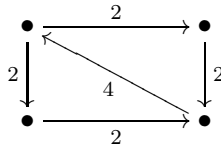
1.4. An example. As an example take $Z = \mathbb{P}^1 \times \mathbb{P}^1$ with its projections $\pi_1, \pi_2: Z \rightarrow \mathbb{P}^1$. We use the standard notation

$$\mathcal{O}(a, b) = \pi_1^* \mathcal{O}_{\mathbb{P}^1}(a) \otimes \pi_2^* \mathcal{O}_{\mathbb{P}^1}(b).$$

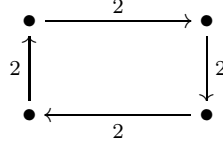
The canonical bundle of Z is $\mathcal{O}(-2, -2)$. The sequence

$$\cdots, \mathcal{O}, \mathcal{O}(1, 0), \mathcal{O}(0, 1), \mathcal{O}(1, 1), \mathcal{O}(2, 2), \cdots$$

is a geometric helix of period 4. The corresponding rolled-up helix algebra has quiver



Tilting at the top right vertex gives another rolled-up helix algebra with quiver



corresponding to the geometric helix

$$\dots, \mathcal{O}, \mathcal{O}(1, 0), \mathcal{O}(1, 1), \mathcal{O}(1, 2), \mathcal{O}(2, 2), \dots$$

Continuing the tilting process one obtains an infinite web of CY_3 algebras, part of which is shown in Figure 1, with the associated quivers.

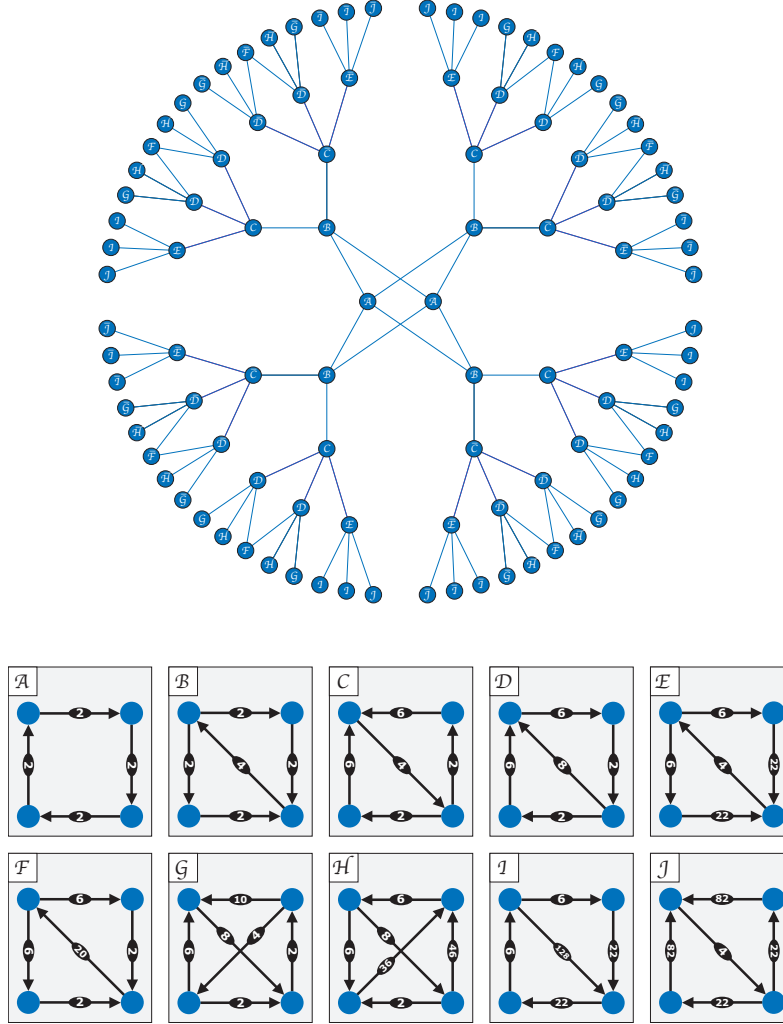


FIGURE 1. A web of Calabi-Yau algebras.

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2. EXCEPTIONAL COLLECTIONS AND MUTATION FUNCTORS

In this section we review the basic definitions and results concerning exceptional collections and mutations following [4, 5]. This material is by-now standard, but we have departed slightly from the usual convention in defining mutation functors using the natural categorical shifts. This eliminates unnecessary shifts from several formulae.

2.1. Assumptions. Throughout \mathcal{D} will be a fixed \mathbb{C} -linear triangulated category. We always assume that

- \mathcal{D} is of *finite type*, i.e. for any two objects A, B the vector space

$$\bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{D}}^i(A, B)$$

is finite-dimensional.

- \mathcal{D} is *algebraic* in the sense of Keller [18, Section 3.6].

The main examples we have in mind are

- The bounded derived category of coherent sheaves $\mathcal{D}(Z)$ on a smooth projective variety Z .
- The bounded derived category of finitely generated A -modules $\mathcal{D}(A)$ over a finite-dimensional algebra A of finite global dimension.

It will usually also be the case that

- \mathcal{D} is *saturated* [5], i.e. all triangulated functors

$$\mathcal{D} \rightarrow \mathcal{D}(\mathbb{C}), \quad \mathcal{D}^{\mathrm{op}} \rightarrow \mathcal{D}(\mathbb{C}),$$

are representable.

Note that if \mathcal{D} has a full exceptional collection then this condition is automatic (see [5, Corollary to Theorem 2.10]). Moreover the two classes of examples (a) and (b) above are saturated by [7, Theorem 1.1] and [5, Theorem 2.11] respectively.

2.2. Exceptional collections. An object $E \in \mathcal{D}$ is said to be *exceptional* if

$$\mathrm{Hom}_{\mathcal{D}}^k(E, E) = \begin{cases} \mathbb{C} & \text{if } k = 0, \\ 0 & \text{otherwise.} \end{cases}$$

An *exceptional collection* $\mathbb{E} \subset \mathcal{D}$ is a sequence of exceptional objects

$$\mathbb{E} = (E_1, \dots, E_n)$$

such that if $1 \leq i < j \leq n$ then $\mathrm{Hom}_{\mathcal{D}}^{\bullet}(E_j, E_i) = 0$.

Given an exceptional collection \mathbb{E} the right orthogonal subcategory to \mathbb{E} is the full triangulated subcategory

$$\mathbb{E}^{\perp} = \{X \in \mathcal{D} : \mathrm{Hom}_{\mathcal{D}}^{\bullet}(E, X) = 0 \text{ for } E \in \mathbb{E}\}.$$

Similarly, the left orthogonal subcategory to \mathbb{E} is

$${}^{\perp}\mathbb{E} = \{X \in \mathcal{D} : \mathrm{Hom}_{\mathcal{D}}^{\bullet}(X, E) = 0 \text{ for } E \in \mathbb{E}\}.$$

When \mathbb{E} consists of a single object E we just write E^{\perp} and ${}^{\perp}E$.

We write $\langle \mathbb{E} \rangle \subset \mathcal{D}$ for the smallest full triangulated subcategory of \mathcal{D} containing the elements of an exceptional collection $\mathbb{E} \subset \mathcal{D}$. An exceptional collection $\mathbb{E} \subset \mathcal{D}$ is said to be *full* if $\langle \mathbb{E} \rangle = \mathcal{D}$. By Lemma C.1 this is equivalent to assuming either that $\mathbb{E}^{\perp} = 0$ or that ${}^{\perp}\mathbb{E} = 0$.

Example 2.1. We give examples for $\mathcal{D} = \mathcal{D}(Z)$ with Z a projective variety.

(a) The sequence

$$(\mathcal{O}, \mathcal{O}(1), \dots, \mathcal{O}(k))$$

is a full exceptional collection on $Z = \mathbb{P}^k$.

(b) The sequence

$$(\mathcal{O}, \mathcal{O}(1, 0), \mathcal{O}(0, 1), \mathcal{O}(1, 1))$$

is a full exceptional collection on $Z = \mathbb{P}^1 \times \mathbb{P}^1$.

(c) The sequence

$$(\mathcal{O}, \mathcal{O}(1, 0), \mathcal{O}(-2, 1), \mathcal{O}(-1, 1))$$

is another full exceptional collection on $Z = \mathbb{P}^1 \times \mathbb{P}^1$.

It is easy to see (see Lemma C.3) that the classes of the elements of a full exceptional collection form a basis for the Grothendieck group $K(D)$. Thus the length of a full exceptional collection (if one exists) is an invariant of the category D .

2.3. Mutation functors. Suppose $E \in D$ is an exceptional. Given an object $X \in {}^\perp E$ the *left mutation of X through E* is the object $L_E(X) \in E^\perp$ defined up to isomorphism by the triangle

$$(5) \quad \mathrm{Hom}_D^\bullet(E, X) \otimes E \xrightarrow{ev} X \longrightarrow L_E(X),$$

where ev denotes the evaluation map. Similarly, given $X \in E^\perp$, the *right mutation of X through E* is the object $R_E(X) \in {}^\perp E$ defined by the triangle

$$(6) \quad R_E(X) \longrightarrow X \xrightarrow{coev} \mathrm{Hom}_D^\bullet(X, E)^* \otimes E,$$

where $coev$ denotes the coevaluation map. It is easy to check that these two operations define mutually inverse equivalences of categories

$$\begin{array}{ccc} & L_E & \\ {}^\perp E & \xrightarrow{\quad} & E^\perp \\ & R_E & \end{array}$$

We also consider mutations through exceptional collections. Suppose $\mathbb{E} = (E_1, \dots, E_k)$ is an exceptional collection in D . Define the left mutation of an object $X \in {}^\perp \mathbb{E}$ through the collection \mathbb{E} to be the object

$$L_{\mathbb{E}}(X) = L_{E_1} \cdots L_{E_k}(X) \in \mathbb{E}^\perp.$$

Similarly, define the right mutation of $X \in \mathbb{E}^\perp$ through the collection \mathbb{E} to be the object

$$R_{\mathbb{E}}(X) = R_{E_k} \cdots R_{E_1}(X) \in {}^\perp \mathbb{E}.$$

Once again these two operations define mutually inverse functors

$$\begin{array}{ccc} & L_{\mathbb{E}} & \\ {}^\perp \mathbb{E} & \xrightarrow{\quad} & \mathbb{E}^\perp \\ & R_{\mathbb{E}} & \end{array}$$

Note that passing from the category D to its opposite category D^{op} exchanges left and right orthogonal subcategories and also left and right mutation functors. This symmetry between left and right will be a constant feature in this paper, and we will often make statements just for left mutations, safe in the knowledge that the corresponding statements for right mutations can be deduced by passing to D^{op} .

Remarks 2.2. (a) If (E, F) is an exceptional pair, then

$$\langle (L_E(F), E) \rangle = \langle (E, F) \rangle = \langle (F, R_F(E)) \rangle.$$

This is easily checked directly from the definition.

- (b) There is a more categorical approach to the mutation functors which we briefly recall in Appendix B. From this approach one obtains the following characterisation of mutations. Suppose

$$E \longrightarrow X \longrightarrow Y$$

is a triangle in \mathcal{D} such that $E \in \langle \mathbb{E} \rangle$, $X \in {}^\perp \mathbb{E}$ and $Y \in \mathbb{E}^\perp$. Then it follows that $Y = L_{\mathbb{E}}(X)$ and $X = R_{\mathbb{E}}(Y)$.

- (c) In particular it follows from (b) that the functors $L_{\mathbb{E}}$ and $R_{\mathbb{E}}$ depend only on the subcategory $\langle \mathbb{E} \rangle$ and not on the particular choice of exceptional collection \mathbb{E} .

2.4. Homomorphism algebra. Let $\mathbb{E} = (E_1, \dots, E_n)$ be an exceptional collection in \mathcal{D} . One can associate to \mathbb{E} a finite-dimensional graded algebra

$$A = A(\mathbb{E}) = \text{End}_{\mathcal{D}}\left(\bigoplus_{i=1}^n E_i\right) = \bigoplus_{k \geq 0} \bigoplus_{j-i=k} \text{Hom}_{\mathcal{D}}(E_i, E_j)$$

called the *homomorphism algebra* of \mathbb{E} .

The degree zero part of A is an n -dimensional semisimple algebra. It follows (see Lemma A.1) that A can be presented as a quiver algebra $A(Q, I)$ for a unique quiver Q . The vertices of Q correspond to the elements of the collection \mathbb{E} , and hence are naturally indexed by the set $\{1, \dots, n\}$. By (1) the number of arrows n_{ij} in Q from vertex i to vertex j is 0 unless $i < j$ in which case n_{ij} is the dimension of the cokernel of the map

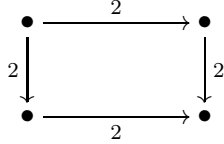
$$\bigoplus_{i < k < j} \text{Hom}_{\mathcal{D}}(E_i, E_k) \otimes \text{Hom}_{\mathcal{D}}(E_k, E_j) \longrightarrow \text{Hom}_{\mathcal{D}}(E_i, E_j).$$

The fact that $n_{ij} = 0$ unless $i < j$ implies that A is of global dimension $\leq n$ (see Remark A.3).

Example 2.3. Consider the full exceptional collection

$$\mathbb{E} = (\mathcal{O}, \mathcal{O}(1, 0), \mathcal{O}(0, 1), \mathcal{O}(1, 1))$$

from Example 2.1(b). Using the above rule it is easy to see that the quiver underlying $A(\mathbb{E})$ is



where the numbers on the edges indicate numbers of arrows. We give another way to compute this quiver in Example 2.8 below.

An exceptional collection $\mathbb{E} = (E_1, \dots, E_n)$ is said to be *strong* if for all i, j

$$\text{Hom}_{\mathcal{D}}^k(E_i, E_j) = 0 \text{ unless } k = 0.$$

In Example 2.1 the collections (a) and (b) are strong, but not the collection (c).

Theorem 2.4. Suppose \mathcal{D} is an algebraic triangulated category of finite type. Suppose $\mathbb{E} = (E_1, \dots, E_n)$ is a full, strong exceptional collection in \mathcal{D} and let $A = A(\mathbb{E})$ be its homomorphism algebra. Then there is an equivalence

$$\Phi_{\mathbb{E}}: \mathcal{D}(A) \longrightarrow \mathcal{D}$$

sending the rank one free module A to the object $E = \bigoplus_{i=1}^n E_i$.

Proof. In the case when $D = D(Z)$ for a smooth projective variety Z this result is due to Bondal [4, Theorem 6.2] and Rickard [25]. As stated it follows from general results of Keller, see [18, Theorem 3.8(a)]. \square

2.5. Dual collection. The following simple result will be very important later.

Lemma 2.5. *Let $\mathbb{E} = (E_1, \dots, E_n)$ be a full exceptional collection and define*

$$(7) \quad F_j = L_{E_1} \cdots L_{E_{j-1}}(E_j) \quad 1 \leq j \leq n$$

Then $\mathbb{F} = (F_n, \dots, F_1)$ is a full exceptional collection and

$$(8) \quad \mathrm{Hom}_D^k(E_i, F_j) = \begin{cases} \mathbb{C} & i = j \text{ and } k = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Conversely, if a sequence of objects (F_n, \dots, F_1) satisfy (8) then they are given by (7).

Proof. A special case of this appears in [4, Lemma 5.6]. For the convenience of the reader we give a proof in Appendix B. \square

We shall call the collection \mathbb{F} of Lemma 2.5 the *dual collection* to \mathbb{E} . The importance of dual collections is as follows. Suppose \mathbb{E} is a full, strong exceptional collection and let $A = A(\mathbb{E})$ denote the corresponding homomorphism algebra. Let S_i be the one-dimensional simple A -module corresponding to vertex i , and $P_i = e_i A$ the indecomposable projective module.

Lemma 2.6. *Under the equivalence $\Phi_{\mathbb{E}}$ of Theorem 2.4*

$$\Phi_{\mathbb{E}}(P_i) = E_i, \quad \Phi_{\mathbb{E}}(S_i) = F_i.$$

Proof. The first statement holds because the objects E_i are the indecomposable summands of the object E , and the modules P_i are the indecomposable summands of the module A . The second statement follows from the fact that

$$\mathrm{Hom}_A^k(P_i, S_j) = \begin{cases} \mathbb{C} & i = j \text{ and } k = 0, \\ 0 & \text{otherwise,} \end{cases}$$

together with the uniqueness statement of Lemma 2.5. \square

Remark 2.7. Later we shall often use the following fact. In the situation of Theorem 2.4 the standard t-structure on $D(A)$ corresponds under the equivalence $\Phi_{\mathbb{E}}$ to a bounded, non-degenerate t-structure D . The heart of this t-structure is equivalent to $\mathrm{Mod}(A)$ and is a finite-length abelian category whose simple objects are the elements of the dual collection \mathbb{F} . An element $X \in D$ lies in this heart precisely if $\mathrm{Hom}_D^k(E_i, X) = 0$ for $1 \leq i \leq n$ and $k \neq 0$.

By (2) from the introduction and Lemma 2.6, working out the dimensions of the spaces $\mathrm{Hom}_D^1(F_i, F_j)$ gives another way to compute the quiver underlying the homomorphism algebra of a full, strong exceptional collection.

Example 2.8. Take $Z = \mathbb{P}^1 \times \mathbb{P}^1$ and consider again the full, strong exceptional collection

$$\mathbb{E} = (\mathcal{O}, \mathcal{O}(1, 0), \mathcal{O}(0, 1), \mathcal{O}(1, 1)).$$

The corresponding dual collection \mathbb{F} is

$$\mathbb{F} = (\mathcal{O}(-1, -1)[2], \mathcal{O}(0, -1)[1], \mathcal{O}(-1, 0)[1], \mathcal{O}).$$

Computing the dimensions of the spaces $\mathrm{Hom}_Z^1(F_i, F_j)$ gives the quiver of Example 2.3.

2.6. Serre functor. A Serre functor on D is an autoequivalence S_D of D for which there are natural isomorphisms

$$\mathrm{Hom}_D(A, B) = \mathrm{Hom}_D(B, S_D(A))^*.$$

It is easy to show that if a Serre functor exists then it is unique up to isomorphism. The motivating example is when Z is a smooth projective variety of dimension k . Then

$$S_D(-) = (- \otimes \omega_Z)[k]$$

is a Serre functor on $D = D(Z)$. A saturated triangulated category of finite type always has a Serre functor [5, Corollary 3.5].

Suppose $\mathbb{E} \subset D$ is an exceptional collection, and assume that D is saturated. The categories ${}^\perp \mathbb{E}$ and \mathbb{E}^\perp are then also saturated [5, Prop. 2.8] and hence have Serre functors. Since the mutation functor $L_{\mathbb{E}}: {}^\perp \mathbb{E} \rightarrow \mathbb{E}^\perp$ is an equivalence it must commute with these functors. Thus there is a commutative diagram

$$\begin{array}{ccc} {}^\perp \mathbb{E} & \xrightarrow{L_{\mathbb{E}}} & \mathbb{E}^\perp \\ S_{{}^\perp \mathbb{E}} \downarrow & & \downarrow S_{\mathbb{E}^\perp} \\ {}^\perp \mathbb{E} & \xrightarrow{L_{\mathbb{E}}} & \mathbb{E}^\perp \end{array}$$

On the other hand, the definition of a Serre functor shows that S_D restricts to give an equivalence

$$(9) \quad S_D|_{{}^\perp \mathbb{E}}: {}^\perp \mathbb{E} \rightarrow \mathbb{E}^\perp.$$

The following Lemma shows that this functor is the diagonal of the above square.

Lemma 2.9. *Suppose \mathbb{E} is an exceptional collection in D . Then*

$$S_{\mathbb{E}^\perp} \circ L_{\mathbb{E}} = S_D|_{{}^\perp \mathbb{E}} = L_{\mathbb{E}} \circ S_{{}^\perp \mathbb{E}}.$$

Proof. This is well-known to experts. We give a proof in Appendix B. \square

Corollary 2.10. *If (E_1, \dots, E_n) is a full exceptional collection in D then*

$$S_D(E_n) = L_{E_{n-1}} \cdots L_{E_1}(E_n).$$

Proof. Set $\mathbb{E} = (E_1, \dots, E_{n-1})$. By Lemma C.2 the subcategory ${}^\perp \mathbb{E} \subset D$ is generated by a single exceptional object E_n . It follows that it has trivial Serre functor. Applying Lemma 2.9 gives the result. \square

3. HELICES

Unlike the last section, the material in this section is not entirely standard. First we introduce a new and more flexible definition of a helix which we feel is an improvement on previous definitions. We then consider rolled-up helix algebras, and prove Theorem 1.6 from the introduction.

3.1. The definition. We shall use the following definition of a helix.

Definition 3.1. A sequence of objects $\mathbb{H} = (E_i)_{i \in \mathbb{Z}}$ in D is a *helix* if there exist positive integers (n, d) with $d \geq 2$ such that

- (i) for each $i \in \mathbb{Z}$ the corresponding *thread* $(E_{i+1}, \dots, E_{i+n})$ is a full exceptional collection,
- (ii) for each $i \in \mathbb{Z}$ one has $E_{i-n} = S_D(E_i)[1-d]$.

Note that by condition (i) we may as well assume that D has a full exceptional collection. Then D is saturated and hence has a Serre functor S_D , so condition (ii) makes sense. The reason for the apparently strange choice of shift in (ii) will become clear later.

Remarks 3.2. (a) The pair (n, d) is determined by the helix \mathbb{H} ; indeed n is the rank of $K(D)$ and d is then determined by condition (ii). We say that \mathbb{H} is of type (n, d) .

(b) By Corollary 2.10 condition (ii) is equivalent to the statement that for all $i \in \mathbb{Z}$

$$E_{i-n} = L_{E_{i-(n-1)}} \cdots L_{E_{i-1}}(E_i)[1-d].$$

(c) It follows from Remarks 2.2(a) and 3.2(b) that it is enough to check that a single thread of \mathbb{H} is a full exceptional collection.

(d) A full exceptional collection \mathbb{E} of length n generates a helix \mathbb{H} of type (n, d) for each d in the obvious way. Conversely, a helix \mathbb{H} is determined by a single thread $\mathbb{E} \subset \mathbb{H}$ together with the number d .

A helix $\mathbb{H} = (E_i)_{i \in \mathbb{Z}}$ is said to be *geometric* if for all $i < j$

$$\mathrm{Hom}_D^k(E_i, E_j) = 0 \text{ unless } k = 0.$$

It is *strong* if it satisfies the weaker condition that each thread is a strong exceptional collection.

Example 3.3. We give examples when $D = D(Z)$ with Z a projective variety.

(a) Take $Z = \mathbb{P}^{d-1}$. The canonical bundle is $\omega_Z = \mathcal{O}(-d)$. The sequence

$$(\cdots, \mathcal{O}, \mathcal{O}(1), \mathcal{O}(2), \cdots)$$

is a geometric helix of type (d, d) .

(b) Take $Z = \mathbb{P}^1 \times \mathbb{P}^1$. The canonical bundle is $\omega_Z = \mathcal{O}(-2, -2)$. The sequence

$$(\cdots, \mathcal{O}, \mathcal{O}(1, 0), \mathcal{O}(0, 1), \mathcal{O}(1, 1), \mathcal{O}(2, 2), \cdots)$$

is a geometric helix of type $(4, 3)$.

(c) Take $Z = \mathbb{P}^1 \times \mathbb{P}^1$ again. Then

$$(\cdots, \mathcal{O}, \mathcal{O}(1, 0), \mathcal{O}(3, 1), \mathcal{O}(4, 1), \mathcal{O}(2, 2), \cdots)$$

is a non-strong helix of type $(4, 3)$.

In previous treatments helices have usually been defined via the condition in Remark (b), and have been required to be of type (n, n) . This means that perfectly sensible helices such as Examples 3.3(b),(c) are disallowed.

Later we shall need

Lemma 3.4. Suppose \mathbb{H} is a helix of type (n, d) and take a thread (E_1, \cdots, E_n) with dual collection (F_n, \cdots, F_1) . Then the dual collection to the neighbouring thread (E_0, \cdots, E_{n-1}) is

$$(L_{F_n}(F_{n-1}), \cdots, L_{F_n}(F_1), F_n[1-d]).$$

Proof. This is easily proved by directly checking the defining property (8) of the dual collection. Indeed, if $1 \leq i, j \leq n-1$ then $\mathrm{Hom}_D^\bullet(E_i, F_n) = 0$ so

$$\mathrm{Hom}_D^\bullet(E_i, L_{F_n}(F_j)) = \mathrm{Hom}_D^\bullet(E_i, F_j) = \mathbb{C}^{\delta_{ij}}.$$

By Remark 3.2(b) and (7) there is an isomorphism $E_0 = F_n[1-d]$. Since $L_{F_n}(F_j) \in F_n^\perp$ one thus obtains the remaining vanishing. \square

3.2. Rolled-up helix algebras. A helix $\mathbb{H} = (E_i)_{i \in \mathbb{Z}}$ of type (n, d) defines a graded algebra

$$A(\mathbb{H}) = \bigoplus_{k \geq 0} \prod_{j-i=k} \text{Hom}_D(E_i, E_j)$$

known as the *helix algebra*. It has a \mathbb{Z} -action induced by the Serre functor

$$S_D[1-d]: \text{Hom}_D(E_i, E_j) \rightarrow \text{Hom}_D(E_{i-n}, E_{j-n}).$$

The *rolled-up helix algebra* $B(\mathbb{H})$ is defined to be the subalgebra of invariant elements.

Both algebras $A(\mathbb{H})$ and $B(\mathbb{H})$ are graded, and it follows from Lemma A.1 that they are quiver algebras. Given integers $i < j$ let us write a_{ij} for the dimension of the cokernel of the map

$$\bigoplus_{i < k < j} \text{Hom}_D(E_i, E_k) \otimes \text{Hom}_D(E_k, E_j) \longrightarrow \text{Hom}_D(E_i, E_j).$$

If $i \geq j$ we set $a_{ij} = 0$. The quiver underlying $A(\mathbb{H})$ has vertices labelled by the elements of \mathbb{Z} and a_{ij} arrows connecting vertex i to vertex j . The quiver underlying $B(\mathbb{H})$ has vertices corresponding to elements of $\mathbb{Z}/n\mathbb{Z}$ and

$$n_{ij} = \sum_{p \in \mathbb{Z}} a_{i, j+pn}$$

arrows connecting vertex i to vertex j .

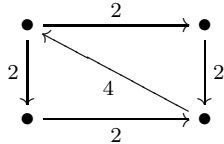
Given a thread $\mathbb{E} = (E_{j+1}, \dots, E_{j+n}) \subset \mathbb{H}$ there is a bijection between the vertices of the quivers underlying $A(\mathbb{E})$ and $B(\mathbb{H})$, which sends the vertex of $A(\mathbb{E})$ corresponding to an object $E_i \in \mathbb{E}$ to the vertex of $B(\mathbb{H})$ labelled by $i \in \mathbb{Z}/n\mathbb{Z}$. It is then easy to see that the quiver for $B(\mathbb{H})$ is obtained from that for $A(\mathbb{E})$ by adding extra arrows corresponding to irreducible morphisms in \mathbb{H} not contained in the thread \mathbb{E} .

We will show later (Prop. 7.5) that in the cases of most interest to us, $a_{ij} = 0$ unless $j - i \leq n$. In that case the extra arrows in $B(\mathbb{H})$ all point backwards with respect to the natural ordering on the vertices of $A(\mathbb{E})$, and, in particular, the quiver for $B(\mathbb{H})$ has no loops.

Example 3.5. Take $Z = \mathbb{P}^1 \times \mathbb{P}^1$. Consider the geometric helix

$$\mathbb{H} = (\dots, \mathcal{O}, \mathcal{O}(1, 0), \mathcal{O}(0, 1), \mathcal{O}(1, 1), \mathcal{O}(2, 2), \dots)$$

of Example 3.3(b). The quiver for the rolled-up helix algebra $B(\mathbb{H})$ is easily seen to be



Comparing with Example 2.3 we see that the quiver for $B(\mathbb{H})$ is obtained from the quiver for the homomorphism algebra of the thread

$$\mathbb{E} = (\mathcal{O}, \mathcal{O}(1, 0), \mathcal{O}(0, 1), \mathcal{O}(1, 1))$$

by adding extra arrows corresponding to irreducible morphisms in \mathbb{H} not contained in \mathbb{E} .

3.3. Geometric interpretation. Suppose now that $D = D(Z)$ with Z a smooth projective Fano variety. Let Y denote the total space of the canonical line bundle of Z with its bundle map $\pi: Y \rightarrow Z$. Set $n = \text{rank } K(Z)$ and $d = \dim_{\mathbb{C}}(Y)$. The following result is adapted from [8, Prop. 4.1] and [27, Prop. 7.2].

Theorem 3.6. *Let $B = B(\mathbb{H})$ be the rolled-up helix algebra of a geometric helix of type (n, d) on Z . Then B is a graded CY_d quiver algebra which is noetherian and finite over its centre. Given a thread $\mathbb{E} \subset \mathbb{H}$ there is an equivalence*

$$\Phi_{\mathbb{E}}: D(B) \longrightarrow D(Y)$$

sending B to the object $\pi^*(E)$, where $E = \bigoplus_{E_j \in \mathbb{E}} E_j$.

Proof. Without loss of generality we can assume that $\mathbb{E} = (E_1, \dots, E_n)$. The statement that the collection \mathbb{E} is full is equivalent to the statement that E classically generates $D(Z)$ in the sense of [7]. By [7, Theorems 2.1.2 and 3.1.1] this is in turn equivalent to the statement that E generates the category $DQcoh(Z)$. By the adjunction

$$\mathrm{Hom}_Y^\bullet(\pi^*(E), F) = \mathrm{Hom}_Z^\bullet(E, \pi_*(F))$$

and the fact that π_* is affine this implies that $\pi^*(E)$ generates $DQcoh(Y)$. Applying the same argument in reverse this means that $\pi^*(E)$ classically generates $D(Y)$.

Next note that $\pi_*(\mathcal{O}_Y)$ is the sheaf of algebras

$$\mathcal{A} = \pi_*(\mathcal{O}_Y) = \bigoplus_{p \geq 0} \omega_Z^{-p}$$

with the obvious product structure. Since π is affine it is then standard (see [13, Ex. II.5.17]) that there is an equivalence of categories between \mathcal{O}_Y -modules on Y and \mathcal{A} -modules on Z , defined by sending an \mathcal{O}_Y -module M to the \mathcal{O}_Z -module $\pi_*(M)$ together with the induced module structure $\mathcal{A} \otimes \pi_*(M) \rightarrow \pi_*(M)$. The object $\pi^*(E)$ on Y corresponds under this equivalence to

$$\pi_*\pi^*(E) = E \otimes_{\mathcal{O}_Z} \mathcal{A} = \bigoplus_{i \geq 0} E_i$$

with the induced \mathcal{A} -module structure. The assumption that \mathbb{H} is geometric implies that

$$\mathrm{Ext}_Y^p(\pi^*(E), \pi^*(E)) = \mathrm{Ext}_Z^p(E, \pi_*\pi^*(E)) = 0 \text{ for all } p > 0.$$

Thus $\pi^*(E)$ is a classical tilting object in the sense of [16]. The endomorphisms of this object are precisely the rolled-up helix algebra B .

The algebra

$$A = \Gamma(Y, \mathcal{O}_Y) = \bigoplus_{p \geq 0} \Gamma(Z, \omega_Z^{-p}).$$

is finitely-generated by the assumption that ω_Z^{-1} is ample (see [24, Theorem 2.3]). The affine variety $X = \mathrm{Spec}(A)$ is the cone over the variety $Z = \mathrm{Proj}(A)$ and there is a projective morphism $p: Y \rightarrow X$ contracting the zero-section of the bundle $\pi: Y \rightarrow Z$. Applying [16, Theorem 7.6] we conclude that B is finite over its centre and has finite global dimension, and that there is an equivalence as claimed.

Finally the equivalence $\Phi_{\mathbb{E}}$ restricts to an equivalence

$$D_{\mathrm{fin}}(B) \longrightarrow D_c(Y)$$

where $D_c(Y) \subset D(Y)$ is the subcategory of objects with compact supports. The CY_d property then follows from Serre duality on Y . \square

Let Q be the quiver underlying the rolled-up helix algebra $\mathbb{H} = B(\mathbb{H})$. The vertices are in natural bijection with the objects of any thread $\mathbb{E} \subset \mathbb{H}$. Let P_i and S_i denote the projective and simple modules associated to a vertex $i \in Q_0$. The analogue of Lemma 2.6 is

Lemma 3.7. *Under the equivalence $\Phi_{\mathbb{E}}$ of Theorem 3.6 one has*

$$\Phi_{\mathbb{E}}(P_i) = \pi^*(E_i), \quad \Phi_{\mathbb{E}}(S_i) = i_*(F_i)$$

where \mathbb{F} be the dual collection to the thread \mathbb{E} , and $i: Z \hookrightarrow Y$ is the inclusion of the zero-section.

Proof. The adjunction $\pi^* \dashv \pi_*$ and the fact that $\pi \circ i = \text{id}_Z$ gives

$$\text{Hom}_Y(\pi^*(E_i), i_*(F_j)) = \text{Hom}_Z(E_i, F_j).$$

The result then follows as in Lemma 2.6. \square

4. MUTATION OPERATIONS

The word mutation is used to mean two different but related things in the theory of exceptional collections. Mutation functors were defined in Section 2 and are equivalences between certain subcategories of our triangulated category \mathcal{D} . Mutations of exceptional collections on the other hand are operations on the set of all exceptional collections in \mathcal{D} . In this section we focus on mutations in this second sense. The main question that arises is under what circumstances such mutations preserve purity properties of exceptional collections.

4.1. Standard mutations. Consider the set of all exceptional collections in \mathcal{D} of a certain length, say n . We can use mutation functors to define operations on this set. The standard way to do this is to define an operation σ_i for each $1 < i \leq n$ by the rule

$$\sigma_i(E_1, \dots, E_{i-2}, E_{i-1}, E_i, E_{i+1}, \dots, E_n) = (E_1, \dots, E_{i-2}, L_{E_{i-1}}(E_i)[-1], E_{i-1}, E_{i+1}, \dots, E_n).$$

It is easy to check that this operation does indeed take exceptional collections to exceptional collections. By Remark 2.2(a) it also takes full collections to full collections.

Theorem 4.1 (Bondal, Gorodentsev, Rudakov). *The operations $\sigma_2, \dots, \sigma_n$ satisfy the braid relations*

$$\begin{aligned} \sigma_i \circ \sigma_{i+1} \circ \sigma_i &= \sigma_{i+1} \circ \sigma_i \circ \sigma_{i+1} \\ \sigma_i \circ \sigma_j &= \sigma_j \circ \sigma_i \quad \text{if } |j - i| > 1 \end{aligned}$$

and hence generate an action of the n -string braid group Br_n on the set of exceptional collections of length n .

Proof. The second relation is obvious, so we just prove the first. To simplify notation take $n = 3$ and $i = 2$. Then

$$\sigma_3 \circ \sigma_2 \circ \sigma_3(E_1, E_2, E_3) = (L_{\mathbb{F}}(E_3)[-2], L_{E_2}(E_1)[-1], E_2),$$

where $\mathbb{F} = (E_1, E_2)$. On the other hand

$$\sigma_2 \circ \sigma_3 \circ \sigma_2(E_1, E_2, E_3) = (L_{\mathbb{G}}(E_3)[-2], L_{E_2}(E_1)[-1], E_2),$$

where $\mathbb{G} = \sigma_2(\mathbb{F}) = (L_{E_2}(E_1)[-1], E_2)$. Applying Remarks 2.2(a) and (c) gives the required relation. \square

When \mathcal{D} is equipped with a particular choice of t-structure $\mathcal{D}^{\leq 0} \subset \mathcal{D}$ we shall say that an exceptional collection \mathbb{E} or a helix \mathbb{H} is *pure* if each constituent object lies in its heart. The shift of the mutated object $L_{E_{i-1}}(E_i)$ put into the definition of the standard mutation operation σ_i is immaterial for the existence of the braid group action, since the mutation functors commute with shifts. However, the given shift ensures that in certain situations the standard mutations preserve pure collections.

Theorem 4.2 (Bondal, Polishchuk). *Suppose $\mathbb{E} = (E_1, \dots, E_n)$ is a full exceptional collection in \mathcal{D} and take $1 < i \leq n$. Suppose \mathcal{D} is equipped with a t -structure that is preserved by the autoequivalence $S_{\mathcal{D}}[1-n]$. Then*

- (i) \mathbb{E} pure implies $\sigma_i(\mathbb{E})$ pure,
- (ii) \mathbb{E} pure implies \mathbb{E} strong.

Theorem 4.2 will be proved in Section 5 (as a special case of Theorem 4.5). It has the following neat consequence.

Corollary 4.3. *Suppose Z is a smooth projective variety satisfying*

$$\text{rank } K(Z) = \dim(Z) + 1.$$

Then the standard mutation operations σ_i preserve full collections of sheaves, and all such collections are strong.

Proof. Take $\mathcal{D} = \mathcal{D}(Z)$ equipped with the standard t -structure. The length of any full exceptional collection in \mathcal{D} is the rank of the Grothendieck group $K(\mathcal{D}) = K(Z)$. Since $S_{\mathcal{D}}(E) = E \otimes \omega_Z[\dim(Z)]$ the result follows from Theorem 4.2. \square

Corollary 4.3 applies for example if $Z = \mathbb{P}^k$ is a projective space. On other varieties the standard mutation operations need not preserve collections of sheaves (see Example 4.7 below). Our aim will be to develop classes of mutation operations which preserve purity properties in more general situations.

4.2. Block mutations. Recall that two objects E and E' of \mathcal{D} are said to be *orthogonal* if

$$\text{Hom}_{\mathcal{D}}^{\bullet}(E, E') = 0 = \text{Hom}_{\mathcal{D}}^{\bullet}(E', E).$$

It is possible to generalise Theorem 4.2 by considering exceptional collections which can be split up into blocks of mutually orthogonal objects. Such collections were studied by Hille [15].

Definition 4.4. A d -block exceptional collection is an exceptional collection together with a partition of \mathbb{E} into d subcollections

$$\mathbb{E} = (\mathbb{E}_1, \dots, \mathbb{E}_d),$$

called blocks, such that the objects in each block \mathbb{E}_i are mutually orthogonal.

For each integer $1 < i \leq d$ we can define an operation τ_i on d -block collections in \mathcal{D} by the rule

$$\tau_i(\mathbb{E}_1, \dots, \mathbb{E}_{i-2}, \mathbb{E}_{i-1}, \mathbb{E}_i, \mathbb{E}_{i+1}, \dots, \mathbb{E}_d) = (\mathbb{E}_1, \dots, \mathbb{E}_{i-2}, L_{\mathbb{E}_{i-1}}(\mathbb{E}_i)[-1], \mathbb{E}_{i-1}, \mathbb{E}_{i+1}, \dots, \mathbb{E}_d).$$

Here, if $\mathbb{E}_i = (E_{a+1}, \dots, E_b)$ then by definition

$$L_{\mathbb{E}_{i-1}}(\mathbb{E}_i) = (L_{\mathbb{E}_{i-1}}(E_{a+1}), \dots, L_{\mathbb{E}_{i-1}}(E_b)).$$

The same proof as before shows that these operations define an action of the group Br_d on the set of d -block exceptional collections.

Theorem 4.5. *Suppose $\mathbb{E} = (\mathbb{E}_1, \dots, \mathbb{E}_d)$ is a full d -block collection and take $1 < i \leq d$. Suppose \mathcal{D} is equipped with a t -structure that is preserved by the autoequivalence $S_{\mathcal{D}}[1-d]$. Then*

- (i) \mathbb{E} pure implies $\tau_i(\mathbb{E})$ pure,
- (ii) \mathbb{E} pure implies \mathbb{E} strong.

We give the proof of Theorem 4.5 in Section 5. Note that Theorem 4.2 is a special case of Theorem 4.5 since any full exceptional collection of length n is automatically an n -block collection, and the operations τ_i then agree with the standard mutations σ_i .

The same argument we gave for Corollary 4.3 gives

Corollary 4.6. *Suppose Z is a smooth projective variety of dimension $d - 1$. Then the d -block mutation operations τ_i preserve full d -block collections of sheaves, and all such collections are strong.* \square

Example 4.7. Consider the full, strong exceptional collection of Example 2.1(b) on $Z = \mathbb{P}^1 \times \mathbb{P}^1$. Then

$$\sigma_3(\mathcal{O}, \mathcal{O}(1, 0), \mathcal{O}(0, 1), \mathcal{O}(1, 1)) = (\mathcal{O}, \mathcal{O}(0, 1)[-1], \mathcal{O}(1, 0), \mathcal{O}(1, 1))$$

which does not consist of sheaves, and

$$\sigma_4(\mathcal{O}, \mathcal{O}(1, 0), \mathcal{O}(0, 1), \mathcal{O}(1, 1)) = (\mathcal{O}, \mathcal{O}(1, 0), \mathcal{O}(-1, 1), \mathcal{O}(0, 1))$$

which consists of sheaves, but is not strong. If we instead consider the collection as a 3-block collection we have

$$\tau_2(\mathcal{O}, [\mathcal{O}(1, 0), \mathcal{O}(0, 1)], \mathcal{O}(1, 1)) = ([\mathcal{O}(-1, 0), \mathcal{O}(0, -1)], \mathcal{O}, \mathcal{O}(1, 1))$$

which in accordance with Corollary 4.6 is strong and consists of sheaves.

4.3. Mutations of helices. There are also mutation operations on the set of helices in \mathcal{D} of a fixed type (n, d) . Given such a helix $\mathbb{H} = (E_j)_{j \in \mathbb{Z}}$ and an integer i , we define a new helix $\sigma_i(\mathbb{H}) = \mathbb{H}' = (E'_j)_{j \in \mathbb{Z}}$ by the rule

$$E'_j = \begin{cases} E_{j-1} & \text{if } j = i \pmod{n}, \\ L_{E_j}(E_{j+1})[-1] & \text{if } j = i - 1 \pmod{n}, \\ E_j & \text{otherwise.} \end{cases}$$

Note that \mathbb{H}' satisfies the periodicity property in the definition of a helix because the Serre functor is an equivalence and hence commutes with mutations. If $j - i \neq -1 \pmod{n}$ then

$$(E'_{j+1}, \dots, E'_{i-1}, E'_i, \dots, E'_{j+n}) = (E_{j+1}, \dots, L_{E_{i-1}}(E_i)[-1], E_{i-1}, \dots, E_{j+n})$$

Thus \mathbb{H}' can be obtained by mutating any thread of \mathbb{H} containing the objects E_{i-1} and E_i and then taking the corresponding helix. The threads of the mutated helix are full by Remarks 2.2(a) and 3.2(c).

The periodicity property ensures that the operation σ_i only depends on the class of i modulo n . There is another natural operation ρ on helices defined by turning the screw

$$\rho[(E_i)_{i \in \mathbb{Z}}] = (E_{i+1})_{i \in \mathbb{Z}}.$$

Theorem 4.1 immediately implies the following result.

Theorem 4.8. *The operations ρ and σ_i for $i \in \mathbb{Z}/n\mathbb{Z}$ satisfy the relations*

$$\begin{aligned} \rho \circ \sigma_i &= \sigma_{i+1} \circ \rho \\ \sigma_i \circ \sigma_{i+1} \circ \sigma_i &= \sigma_{i+1} \circ \sigma_i \circ \sigma_{i+1} \\ \sigma_i \circ \sigma_j &= \sigma_j \circ \sigma_i \quad \text{if } j - i \neq \pm 1 \end{aligned}$$

and hence define an action of the affine braid group $\widehat{\text{Br}}_n$ on the set of helices of type (n, d) . \square

Just as the usual n -string braid group Br_n is the fundamental group of the configuration space of n points in \mathbb{C} , the affine braid group occurring in Theorem 4.8 is the fundamental group of the configuration space of n points in \mathbb{C}^* (see [21]).

4.4. Levelled mutations. Finally in this section we introduce the notion of a levelling, which will allow us to keep track of operations defined by mutations through subcollections.

Definition 4.9. A *levelling* on an exceptional collection $\mathbb{E} = (E_1, \dots, E_n)$ is a function $\phi: \mathbb{E} \rightarrow \mathbb{Z}$ such that $i \leq j \implies \phi(E_i) \leq \phi(E_j)$.

A levelling on an exceptional collection \mathbb{E} partitions \mathbb{E} into disjoint subcollections $\mathbb{E}_i = \phi^{-1}(i)$. We refer to these subcollections as levels. Note that we do not assume that the objects in a given level are orthogonal.

There are mutation operations σ_i on pairs (\mathbb{E}, ϕ) consisting of an exceptional collection \mathbb{E} and a levelling $\phi: \mathbb{E} \rightarrow \mathbb{Z}$. Given $i \in \mathbb{Z}$ we define the mutation $\sigma_i(\mathbb{E}, \phi)$ of the pair (\mathbb{E}, ϕ) at level i to be the pair (\mathbb{E}', ϕ') for which the corresponding levels are $\mathbb{E}'_j = \mathbb{E}_j$ for $j \notin \{i-1, i\}$, and

$$\mathbb{E}'_{i-1} = L_{\mathbb{E}_{i-1}}(\mathbb{E}_i)[-1], \quad \mathbb{E}'_i = \mathbb{E}_{i-1}.$$

Suppose for example that $\mathbb{E} = (\mathbb{E}_1, \dots, \mathbb{E}_d)$ is a d -block collection. There is a unique levelling $\phi: \mathbb{E} \rightarrow \mathbb{Z}$ satisfying $\phi^{-1}(i) = \mathbb{E}_i$. We call it the *canonical levelling*. For $1 < i \leq d$ the mutated pair $(\mathbb{E}', \phi') = \sigma_i(\mathbb{E}, \phi)$ is precisely the d -block collection $\tau_i(\mathbb{E})$ together with its canonical levelling.

We also consider levellings on helices.

Definition 4.10. A levelling on a helix $\mathbb{H} = (E_i)_{i \in \mathbb{Z}}$ of type (n, d) is a function $\phi: \mathbb{H} \rightarrow \mathbb{Z}$ such that

- (i) $i \leq j \implies \phi(E_i) \leq \phi(E_j)$,
- (ii) for each $i \in \mathbb{Z}$ one has $\phi(E_{i+n}) = \phi(E_i) + d$.

There are mutation operations σ_i defined on pairs (\mathbb{H}, ϕ) consisting of a helix \mathbb{H} of type (n, d) and a levelling $\phi: \mathbb{H} \rightarrow \mathbb{Z}$. The levelling ϕ partitions \mathbb{H} into exceptional collections $\mathbb{E}_j = \phi^{-1}(j)$. To define $\sigma_i(\mathbb{H}, \phi)$ take a thread \mathbb{E} of \mathbb{H} containing the levels \mathbb{E}_i and \mathbb{E}_{i-1} . This can always be done by the periodicity of ϕ and our assumption that $d \geq 2$ in the definition of a helix. Since ϕ restricts to a levelling on \mathbb{E} , we can define the mutated pair

$$(\mathbb{E}', \phi') = \sigma_i(\mathbb{E}, \phi)$$

and then set $\sigma_i(\mathbb{H}, \phi)$ to be the helix generated by \mathbb{E}' , together with the unique levelling extending ϕ' . It is easy to check that the resulting mutated helix $\sigma_i(\mathbb{H}, \phi)$ is independent of the thread \mathbb{E} we chose, and depends only on the value of i modulo d .

5. AN ARGUMENT OF BONDAL AND POLISHCHUK

In this section we give a proof of Theorem 4.5. This involves extending several arguments from [4, 6] to the d -block case. We will later use a similar argument to prove our main result Theorem 7.4. The crucial ingredient in the proofs of these theorems is the following simple observation from [4].

Lemma 5.1. *Let (\mathbb{H}, ϕ) be a pair of a helix \mathbb{H} of type (n, d) together with a levelling $\phi: \mathbb{H} \rightarrow \mathbb{Z}$. Suppose also that the objects of $\mathbb{E}_m = \phi^{-1}(m)$ are mutually orthogonal. Then*

$$(\sigma_{m-(d-2)} \circ \dots \circ \sigma_m)(\mathbb{H}, \phi) = (\mathbb{H}, \phi + 1).$$

Proof. Adding a constant to the levelling we may as well assume that $m = d$. Consider the thread $\mathbb{E} = (\mathbb{E}_1, \dots, \mathbb{E}_{d-1}, \mathbb{E}_d)$ and also the subcollection $\mathbb{E}' = (\mathbb{E}_1, \dots, \mathbb{E}_{d-1})$. Applying the mutations in the statement of the Lemma to the thread \mathbb{E} we obtain the collection

$$(L_{\mathbb{E}'}(\mathbb{E}_d)[1-d], \mathbb{E}_1, \dots, \mathbb{E}_{d-1}).$$

Lemma C.2 implies that ${}^\perp \mathbb{E}' = \langle \mathbb{E}_d \rangle$ and this category has trivial Serre functor by the assumption that all objects of the collection \mathbb{E}_d are orthogonal. Thus by Lemma 2.9 and the periodicity of the helix

$$L_{\mathbb{E}'}(\mathbb{E}_d)[1-d] = S_D(\mathbb{E}_d)[1-d] = \mathbb{E}_0.$$

The mutated thread is therefore

$$(\mathbb{E}_0, \mathbb{E}_1, \dots, \mathbb{E}_{d-1})$$

and so the helix it generates is \mathbb{H} . The induced levelling is easily seen to be $\phi + 1$. \square

A simple consequence is

Lemma 5.2. *Let (\mathbb{H}, ϕ) be a pair of a helix \mathbb{H} of type (n, d) together with a levelling $\phi: \mathbb{H} \rightarrow \mathbb{Z}$. Suppose also that the objects of $\mathbb{E}_d = \phi^{-1}(d)$ are mutually orthogonal. Then for any integer $1 \leq k \leq d$*

$$L_{\mathbb{E}_k} \cdots L_{\mathbb{E}_{d-1}}(\mathbb{E}_d)[k-d] = R_{\mathbb{E}_{k-1}} \cdots R_{\mathbb{E}_1}(\mathbb{E}_0)[k-1]$$

Proof. Note that Lemma 5.1 with $m = d$ implies

$$(10) \quad (\sigma_{k+1} \circ \cdots \circ \sigma_d)(\mathbb{H}, \phi) = (\sigma_2 \circ \cdots \circ \sigma_k)^{-1}(\mathbb{H}, \phi - 1).$$

Now the inverses of the mutation operations σ_i can be written in terms of right mutations in the obvious way. Taking the k th level of both sides of (10) then gives the result. \square

Define a block structure on a helix \mathbb{H} to be a levelling $\phi: \mathbb{H} \rightarrow \mathbb{Z}$ such that any two objects in the same level $\phi^{-1}(m)$ are orthogonal. Clearly, the (n, d) helix \mathbb{H} generated by a d -block exceptional collection \mathbb{E} has a block structure defined by extending the canonical levelling from \mathbb{E} to \mathbb{H} . Conversely, a block structure on a helix of type (n, d) partitions each thread of \mathbb{H} into at most $d + 1$ blocks.

Theorem 5.3. *Let $D^{\leq 0} \subset D$ be a t -structure and suppose \mathbb{H} is a helix in D with a block structure $\phi: \mathbb{H} \rightarrow \mathbb{Z}$. Take an integer $m \in \mathbb{Z}$ and set $(\mathbb{H}', \phi') = \sigma_m(\mathbb{H}, \phi)$. Then*

- (i) \mathbb{H} pure implies \mathbb{H}' pure,
- (ii) \mathbb{H} pure implies \mathbb{H} geometric,
- (iii) \mathbb{H} geometric implies \mathbb{H}' geometric.

Proof. Adding a constant to the levelling we may as well assume that $m = d$. Suppose first that \mathbb{H} is pure and consider the thread

$$\mathbb{E} = (\mathbb{E}_1, \dots, \mathbb{E}_{d-1}, \mathbb{E}_d).$$

By periodicity, to prove (i) it will be enough to show that the mutated thread

$$\tau_d(\mathbb{E}) = (\mathbb{E}_1, \dots, L_{\mathbb{E}_{d-1}}(\mathbb{E}_d)[-1], \mathbb{E}_{d-1})$$

is pure. In fact we prove more, namely that for $0 \leq k < d$ the multiply mutated thread

$$\mathbb{E}' = \tau_k \cdots \tau_d(\mathbb{E}) = (\mathbb{E}_1, \dots, \mathbb{E}_{k-1}, L_{\mathbb{E}_k} \cdots L_{\mathbb{E}_{d-1}}(\mathbb{E}_d)[k-d], \mathbb{E}_k, \dots, \mathbb{E}_{d-1})$$

is pure.

Consider the mutation functor $L_{\mathbb{E}_j}$ applied to an element $X \in {}^\perp \mathbb{E}_j$. The assumption that all objects of \mathbb{E}_j are orthogonal implies that there is a triangle

$$(11) \quad L_{\mathbb{E}_j}(X)[-1] \longrightarrow \bigoplus_{E \in \mathbb{E}_j} \mathrm{Hom}_{\mathbf{D}}^\bullet(E, X) \otimes E \longrightarrow X.$$

This can be seen by writing the functor $L_{\mathbb{E}_j}$ as a repeated mutation and using the octahedral axiom. Suppose now that $X \in \mathbf{D}^{\geq 0}$. Since \mathbb{E} is pure, the graded vector space $\mathrm{Hom}_{\mathbf{D}}^\bullet(E, X)$ is concentrated in non-negative degrees, and so using the triangle (11) we can conclude that $L_{\mathbb{E}_j}(X)[-1] \in \mathbf{D}^{\geq 0}$ also.

Applying this result repeatedly shows that $\mathbb{E}' \subset \mathbf{D}^{\geq 0}$. Now note that by Lemma 5.2 the thread \mathbb{E}' can also be written as a repeated right mutation. A similar argument to the above then gives $\mathbb{E}' \subset \mathbf{D}^{\leq 0}$ and hence \mathbb{E}' is pure as claimed.

To prove (ii) assume \mathbb{H} is pure and take a thread $\mathbb{E} \subset \mathbb{H}$ that is partitioned by ϕ into d blocks. We claim that \mathbb{E} is strong. It then follows from Proposition 5.4 below that \mathbb{H} is geometric. To prove the claim suppose that there are elements E_i and E_j of \mathbb{E} with $\mathrm{Hom}_{\mathbf{D}}^k(E_i, E_j) \neq 0$ for some $k > 0$. By applying block mutations we can move blocks between E_i and E_j out of the way and arrive at a collection such that E_i and E_j lie in neighbouring blocks \mathbb{E}_{m-1} and \mathbb{E}_m say. Applying one more mutation gives an exceptional collection with blocks

$$(L_{\mathbb{E}_{m-1}}(\mathbb{E}_m)[-1], \mathbb{E}_{m-1}).$$

By part (i) this collection is pure. But by the triangle (11) above

$$\mathrm{Hom}_{\mathbf{D}}^{-k}(L_{\mathbb{E}_{m-1}}(E_j)[-1], E_i) = \mathrm{Hom}_{\mathbf{D}}^k(E_i, E_j)^* \neq 0$$

giving a contradiction.

To prove (iii) let $\mathbf{D}^{\leq 0} \subset \mathbf{D}$ be the t-structure corresponding to the standard t-structure on $\mathbf{D}(A)$ under the equivalence $\Phi_{\mathbb{E}}$ of Theorem 2.4. By Remark 2.7 the assumption that \mathbb{H} is geometric implies that every object far enough to the right in \mathbb{H} is pure. The argument for part (i) shows that the same is true for the mutated helix \mathbb{H}' . The argument we gave for (ii) then shows that \mathbb{H}' is geometric. \square

Proposition 5.4. *Suppose $\mathbb{E} = (\mathbb{E}_1, \dots, \mathbb{E}_d)$ is a full, strong d -block collection. Let \mathbb{H} be the helix of type (n, d) generated by \mathbb{E} . Then \mathbb{H} is geometric iff*

$$\mathrm{Hom}_{\mathbf{D}}^k(E_i, E_j) = 0 \text{ for all } i < j \text{ and } k < 0.$$

Proof. Let $A = A(\mathbb{E})$ be the homomorphism algebra of \mathbb{E} . Applying Remark 2.7 to the opposite category \mathbf{D}^{op} we see that there is a t-structure $\mathbf{D}^{\leq 0} \subset \mathbf{D}$ whose heart is a finite length abelian category \mathcal{A} consisting of objects X for which $\mathrm{Hom}_{\mathbf{D}}(X, E_i)$ is concentrated in degree 0 for all i . The simple objects (G_n, \dots, G_1) of \mathcal{A} satisfy

$$(12) \quad \mathrm{Hom}_{\mathbf{D}}^k(G_j, E_i) = \begin{cases} \mathbb{C} & i = j \text{ and } k = 0, \\ 0 & \text{otherwise.} \end{cases}$$

It follows from the definition of the Serre functor that $F_j = S_{\mathbf{D}}(G_j)$ where (F_n, \dots, F_1) is the dual collection to \mathbb{E} .

Let τ denote the functor $S_{\mathbf{D}}[1 - d]$. We first claim that τ is left exact with respect to this t-structure, that is that $\tau(\mathbf{D}^{\geq 0}) \subset \mathbf{D}^{\geq 0}$. To prove this it is enough to show that $\tau(G_j) = F_j[1 - d] \in \mathbf{D}^{\geq 0}$ for each j .

By Theorem 5.3(i) the collection

$$(L_{\mathbb{E}_1} \cdots L_{\mathbb{E}_{d-1}}(\mathbb{E}_d)[1 - d], \dots, L_{\mathbb{E}_1}(\mathbb{E}_2)[-1], \mathbb{E}_1)$$

is pure. Note that if X and Y are orthogonal $L_X(Y) = Y$. By (7) it follows that the dual collection (F_n, \dots, F_1) is a reordering of the collection

$$(L_{\mathbb{E}_1} \cdots L_{\mathbb{E}_{d-1}}(\mathbb{E}_d), \dots, L_{\mathbb{E}_1}(\mathbb{E}_2), \mathbb{E}_1).$$

Hence $F_j[1-d]$ lies in $D^{\geq 0}$ for all j .

Now suppose the condition of vanishing of negative Hom groups in \mathbb{H} holds. By periodicity of \mathbb{H} , to prove that \mathbb{H} is geometric it will be enough to show that for $E, E' \in \mathbb{E}$ and all $m \geq 0$ one has

$$\mathrm{Hom}_{\mathbb{D}}^k(\tau^m(E'), E) = 0 \text{ for } k > 0.$$

Since E is an injective object in the heart of $D^{\leq 0}$ and $\tau^m(E') \in D^{\geq 0}$ the result follows. \square

Proof of Theorem 4.5. Since the t-structure is preserved by the functor $S_{\mathbb{D}}[1-d]$, the helix \mathbb{H} of type (n, d) generated by \mathbb{E} is pure. The canonical levelling on the d -block collection \mathbb{E} extends to a block structure $\phi: \mathbb{H} \rightarrow \mathbb{Z}$. Thus (i) follows from Theorem 5.3(i). Part (ii) follows as in the proof of Theorem 5.3(ii). \square

6. HEIGHT FUNCTIONS

In this section we introduce special types of levellings called height functions. In the next section we will relate mutations defined by such levellings to the tilting operation for homomorphism and rolled-up helix algebras.

6.1. Relatedness. It will be convenient to introduce the following terminology.

Definition 6.1. Let $\mathbb{E} = (E_1, \dots, E_n)$ be a full exceptional collection with dual collection $\mathbb{F} = (F_n, \dots, F_1)$. Let $p \geq 0$. We say that objects $E_i, E_j \in \mathbb{E}$ with $i \leq j$ are p -related in \mathbb{E} if

$$\mathrm{Hom}_{\mathbb{D}}^k(F_j, F_i) = 0 \text{ for } k \neq p.$$

Note that if F_j and F_i are orthogonal then E_i and E_j are p -related for all p .

A simple Corollary of Theorem 4.5 is

Lemma 6.2. Suppose $\mathbb{E} = (\mathbb{E}_1, \dots, \mathbb{E}_d)$ is a full d -block collection which is pure in some t-structure preserved by $S_{\mathbb{D}}[1-d]$. Let $\phi: \mathbb{E} \rightarrow \mathbb{Z}$ be the canonical levelling. Then for any $i \leq j$ the objects E_i and E_j are $(\phi(j) - \phi(i))$ -related.

Proof. By Theorem 4.5 the collection \mathbb{E} is strong. As we observed in the proof of Proposition 5.4 the collection

$$\mathbb{E}' = (L_{\mathbb{E}_1} \cdots L_{\mathbb{E}_{d-1}}(\mathbb{E}_d)[1-d], \dots, L_{\mathbb{E}_1}(\mathbb{E}_2)[-1], \mathbb{E}_1)$$

is strong, and differs from the dual collection \mathbb{F} by reordering and the shifts. The result follows. \square

Later we shall need

Lemma 6.3. Let $\mathbb{H} = (E_i)_{i \in \mathbb{Z}}$ be a helix of type (n, d) .

- (a) Suppose $i \leq j$ and take two threads $\mathbb{E}_0, \mathbb{E}_1 \subset \mathbb{H}$ containing the objects E_i and E_j . Then E_i and E_j are p -related in the thread \mathbb{E}_0 precisely if the same is true in \mathbb{E}_1 .
- (b) Consider the threads $\mathbb{E}_0 = (E_0, \dots, E_{n-1})$ and $\mathbb{E}_1 = (E_1, \dots, E_n)$ and take an integer $1 \leq i \leq n-1$. Then E_0 and E_i are p -related in \mathbb{E}_0 precisely if E_i and E_n are $(d-p)$ -related in \mathbb{E}_1 .

Proof. For (a) it is enough to consider the case when

$$\mathbb{E}_0 = (E_0, \dots, E_{n-1}), \quad \mathbb{E}_1 = (E_1, \dots, E_n),$$

and $1 \leq i \leq j \leq n-1$. Write $\mathbb{F}_1 = (F_n, \dots, F_1)$ for the dual collection to \mathbb{E}_1 . By Lemma 3.4 the dual collection to \mathbb{E}_0 is

$$\mathbb{F}_0 = (L_{F_n}(F_{n-1}), \dots, L_{F_n}(F_1), F_n[1-d]).$$

Now the dual objects to E_i and E_j in \mathbb{F}_0 and in \mathbb{F}_1 differ by mutation by F_n , and since the mutation functor defines an equivalence ${}^\perp F_n \rightarrow F_n^\perp$, the notion of p -related is the same in each case.

For (b) note that as elements of \mathbb{E}_1 , the dual of E_i is F_i , and the dual of E_n is F_n . On the other hand, as elements of \mathbb{E}_0 the dual of E_0 is $E_0 = F_n[1-d]$ and the dual of E_i is $L_{F_n}(F_i)$. Applying the functor $\mathrm{Hom}_D(-, F_n)$ to the triangle (5) shows that

$$\mathrm{Hom}_D^{-k}(F_n, F_i)^* = \mathrm{Hom}_D^{k+1}(L_{F_n}(F_i), F_n)$$

and the result follows. \square

6.2. Height functions. We now make the following crucial definition.

Definition 6.4. Let $\mathbb{E} = (E_1, \dots, E_n)$ be a full, strong exceptional collection. A levelling $\phi: \mathbb{E} \rightarrow \mathbb{Z}$ is said to be *tilting at level m* if the following condition holds. Suppose $E_i, E_j \in \mathbb{E}$ with $\phi(E_i) = m$ and set $\phi(E_j) = p$. Then

- (i) $i \leq j$ implies that E_i and E_j are $(p-m)$ -related,
- (ii) $j \leq i$ implies that E_j and E_i are $(m-p)$ -related.

A *height function* for an object $E \in \mathbb{E}$ is a levelling $\phi: \mathbb{E} \rightarrow \mathbb{Z}$ which is tilting at level 0 and which satisfies $\phi^{-1}(0) = \{E\}$.

In general a height function for an object E in an exceptional collection \mathbb{E} may or may not exist, and will usually not be in any sense unique.

Examples 6.5. (a) If Z is a smooth projective variety of dimension $d-1$, and \mathbb{E} is a full d -block collection of sheaves on Z then by Lemma 6.2 the canonical levelling on \mathbb{E} is tilting at all levels. It follows from Lemma 6.7 that height functions exist for all objects of \mathbb{E} .

(b) Consider the full, strong collection

$$\mathbb{E} = (\mathcal{O}, \mathcal{O}(1,0), \mathcal{O}(1,1), \mathcal{O}(2,1))$$

on $Z = \mathbb{P}^1 \times \mathbb{P}^1$ with dual collection

$$\mathbb{F} = (\mathcal{O}(0,-1)[2], \mathcal{O}(1,-1)[1], \mathcal{O}(-1,0)[1], \mathcal{O}).$$

The possible height functions for $\mathcal{O}(2,1)$ take values $(-2, b, -1, 0)$ for $b \in \{-2, -1\}$. Similarly, the possible height functions for $\mathcal{O}(1,0)$ take values $(-1, 0, 1, a)$ with $a \geq 1$.

(c) In Section 8 we shall adapt an argument of Herzog to show that if \mathbb{E} is a strong exceptional collection on a del Pezzo surface, then it is possible to reorder \mathbb{E} so that height functions exist for all objects $E \in \mathbb{E}$.

Lemma 6.6. Suppose $\mathbb{E} = (E_1, \dots, E_n)$ is a full strong exceptional collection with dual collection $\mathbb{F} = (F_n, \dots, F_1)$. Suppose $\phi: \mathbb{E} \rightarrow \mathbb{Z}$ is a levelling which is tilting at level m , and take distinct objects $E_i, E_j \in \mathbb{E}_m$. Then E_i is orthogonal to E_j , and moreover F_i is orthogonal to F_j .

Proof. By definition of tilting at level m one has

$$\mathrm{Hom}_{\mathbb{D}}^k(F_i, F_j) = 0 = \mathrm{Hom}_{\mathbb{D}}^k(F_j, F_i)$$

unless $k = 0$. But since the collection \mathbb{E} is strong the objects F_i and F_j are identified under the equivalence $\Phi_{\mathbb{E}}$ with simple modules for the homomorphism algebra $A(\mathbb{E})$. It follows that they must be orthogonal.

Consider the homomorphism algebra $A(\mathbb{E})$ and the corresponding quiver Q . Let the vertex of Q corresponding to an object E_p be denoted p . By (2) the number of arrows n_{pq} from vertex p to vertex q is the dimension of the space $\mathrm{Hom}_{\mathbb{D}}^1(F_q, F_p)$. In particular if $n_{pq} > 0$ then $p < q$. On the other hand the space $\mathrm{Hom}_{\mathbb{D}}(E_p, E_q)$ is spanned by paths in the quiver from vertex p to vertex q , modulo relations. By the first part there are no arrows between vertices corresponding to elements of \mathbb{E}_m . Thus there are no paths between the corresponding vertices either. \square

Corollary 6.7. *Suppose $\mathbb{E} = (E_1, \dots, E_n)$ is a full, strong exceptional collection and take an integer $1 \leq i \leq n$. Suppose there is a levelling $\phi: \mathbb{E} \rightarrow \mathbb{Z}$ which is tilting at level $m = \phi(E_i)$. Then there is a height function for $E_i \in \mathbb{E}$.*

Proof. Replacing ϕ by $\phi - m$ we can assume that $m = 0$. If $E_j \in \mathbb{E}_0$ with $j < i$ then by Lemma 6.6 E_j is p -related to E_i for all p . In this case we redefine $\phi(E_j)$ to be -1 . Similarly for $E_j \in \mathbb{E}_0$ with $j > i$ we set $\phi(E_j) = +1$. The resulting levelling is a height function for $E_i \in \mathbb{E}$. \square

6.3. Height functions on helices. In this section we extend the definition of height functions to helices.

Lemma 6.8. *Let \mathbb{H} be a strong helix with a levelling $\phi: \mathbb{H} \rightarrow \mathbb{Z}$. Suppose ϕ restricted to some thread $\mathbb{E} \subset \mathbb{H}$ containing the exceptional collection $\mathbb{E}_m = \phi^{-1}(m) \subset \mathbb{H}$ is tilting at level m . Then ϕ restricted to any such thread is tilting at level m .*

Proof. We can clearly reduce to the case $m = 0$. Take two threads $\mathbb{E}, \mathbb{E}' \subset \mathbb{H}$ containing \mathbb{E}_0 . We must show that if $\phi|_{\mathbb{E}}$ is tilting at level 0 then so is $\phi|_{\mathbb{E}'}$. It is enough to consider the case when \mathbb{E} and \mathbb{E}' are neighbouring threads, so without loss of generality we can take

$$\mathbb{E} = (E_0, \dots, E_{n-1}), \quad \mathbb{E}' = (E_1, \dots, E_n).$$

Assume $\phi|_{\mathbb{E}}$ is tilting at level 0. Take two objects $E_i, E_j \in \mathbb{E}'$ with $\phi(E_i) = 0$ and set $q = \phi(E_j)$. If $j \leq i$ then $q \leq 0$ and we must show that E_j and E_i are $|q|$ -related in \mathbb{E}' . But this follows from Lemma 6.3(a) because both E_i and E_j lie in \mathbb{E} and $\phi|_{\mathbb{E}}$ is tilting at level 0. So suppose $j \geq i$. Then $q \geq 0$ and we must show that E_i and E_j are q -related. If $E_j \in \mathbb{E}$ the same argument applies. The only other possibility is that $j = n$. Since $\phi|_{\mathbb{E}}$ is tilting at level 0 it follows that E_0 and E_i are $(d - q)$ -related. Applying Lemma 6.3(b) we conclude that E_i and E_n are q -related as required. \square

Now we can make the following definition.

Definition 6.9. Let \mathbb{H} be a strong helix. A levelling $\phi: \mathbb{H} \rightarrow \mathbb{Z}$ is *tilting at level m* if there is a thread $\mathbb{E} \subset \mathbb{H}$ containing $\mathbb{E}_m = \phi^{-1}(m)$ such that the restriction of ϕ to \mathbb{E} is tilting at level m .

A *height function* for an object $E \in \mathbb{H}$ is a levelling $\phi: \mathbb{H} \rightarrow \mathbb{Z}$ which is tilting at level 0 and satisfies $\phi^{-1}(0) = \{E\}$.

The following result shows that providing one knows that height functions exist on strong exceptional collections in a given category \mathbb{D} then one can always construct them on strong helices.

Lemma 6.10. *Let \mathbb{H} be a strong helix and take an object $E \in \mathbb{H}$. Let $\mathbb{E} \subset \mathbb{H}$ be the thread which has E at its start. Suppose there is a height function $\phi: \mathbb{E} \rightarrow \mathbb{Z}$ for $E \in \mathbb{E}$. Then there is a height function for $E \in \mathbb{H}$.*

Proof. Label the helix so that $E = E_1$ and therefore $\mathbb{E} = (E_1, \dots, E_n)$. By hypothesis there exist height functions $\phi: \mathbb{E} \rightarrow \mathbb{Z}$ for $E \in \mathbb{E}$. Choose one such that $\phi(E_n)$ takes the minimal possible value, say p . We claim that $p \leq d - 1$. Extending ϕ periodically then gives a height function for $E \in \mathbb{H}$.

Let $\mathbb{F} = (F_n, \dots, F_1)$ be the dual collection to \mathbb{E} . Note that $F_1 = E_1$. Take $1 \leq m \leq n$ such that $\text{Hom}_{\mathbb{D}}(E_m, E_{n+1}) \neq 0$ and $\text{Hom}_{\mathbb{D}}(E_j, E_{n+1}) = 0$ for $m < j \leq n$. This is possible because \mathbb{E} is full and \mathbb{H} is strong.

Take $1 \leq j \leq n$. By (7) and Remark 3.2(b) the object $E_{j-n}[d-1]$ is obtained by mutating the object $F_j = L_{E_1} \cdots L_{E_{j-1}}(E_j)$ through the collection (E_{j+1-n}, \dots, E_0) . If $m \leq j \leq n$ these objects have no maps to E_1 by periodicity of \mathbb{H} , so

$$\text{Hom}_{\mathbb{D}}^{\bullet}(E_j, E_{n+1})[1-d] = \text{Hom}_{\mathbb{D}}^{\bullet}(E_{j-n}[d-1], E_1) = \text{Hom}_{\mathbb{D}}^{\bullet}(F_j, F_1).$$

For $m < j$ this implies that F_j is orthogonal to F_1 and the value of $\phi(E_j)$ is unconstrained. Since ϕ is assumed minimal this implies that $\phi(E_j) = \phi(E_m)$ for $j > m$. By assumption $\text{Hom}_{\mathbb{D}}(E_m, E_{n+1})$ is nonzero, so $\text{Hom}_{\mathbb{D}}^{\bullet}(F_m, F_1)$ is nonzero in degree $d-1$. Hence E_1 and E_m are $d-1$ related, and $\phi(E_m) = d-1$. \square

7. TILTING AND MUTATION

This section contains our main results. We relate tilting to mutations defined by height functions. We also show that mutations determined by height functions preserve geometric helices.

7.1. Tilting homomorphism algebras. Let $\mathbb{E} = (E_1, \dots, E_n)$ be a full, strong exceptional collection with homomorphism algebra $A = A(\mathbb{E})$. Let Q be the underlying quiver of $A(\mathbb{E})$, let $i \in Q_0$ denote the vertex corresponding to an object E_j and let S_j be the corresponding simple A -module.

Fix a vertex $i \in Q_0$. By Lemma A.2 there is a projective resolution of the module S_i of the form

$$0 \longrightarrow \bigoplus_{j \in Q_0} P_j^{\oplus d_{j,i}^n} \longrightarrow \cdots \longrightarrow \bigoplus_{j \in Q_0} P_j^{\oplus d_{j,i}^1} \longrightarrow P_i \longrightarrow S_i \longrightarrow 0$$

where $d_{j,i}^p = \dim_{\mathbb{C}} \text{Ext}_A^p(S_i, S_j)$. Given an integer $k \geq 0$ consider the truncated complex

$$(13) \quad 0 \longrightarrow \bigoplus_{j \in Q_0} P_j^{\oplus d_{j,i}^k} \longrightarrow \cdots \longrightarrow \bigoplus_{j \in Q_0} P_j^{\oplus d_{j,i}^1} \longrightarrow P_i \longrightarrow 0$$

situated in degrees $-k$ up to 0. We define $P_i^{(k)}$ to be the corresponding element of $D(A)$.

Proposition 7.1. *Suppose $\phi: \mathbb{E} \rightarrow \mathbb{Z}$ is tilting at level m and $E_i \in \mathbb{E}_m$. Then under the equivalence $\Phi_{\mathbb{E}}$ of Theorem 2.4 one has*

$$\Phi_{\mathbb{E}}(P_i^{(k)}) = L_{\mathbb{E}_{m-k}} \cdots L_{\mathbb{E}_{m-1}}(E_i)$$

for all $k \geq 0$.

Proof. Let $\mathbb{F} = (F_n, \dots, F_1)$ be the dual collection to $\mathbb{E} = (E_1, \dots, E_n)$. Recall from Lemma 2.6 that under the equivalence $\Phi_{\mathbb{E}}$ the projective module P_i corresponds to E_i and the simple module S_i corresponds to F_i . Since ϕ is tilting at level m we have $d_{j,i}^p = 0$ unless $E_j \in \mathbb{E}_{m-p}$.

Let \mathbb{E}' be the subcollection

$$\mathbb{E}' = (\mathbb{E}_{m-k}, \dots, \mathbb{E}_{m-1}) \subset \mathbb{E}$$

and set $R = \Phi_{\mathbb{E}}(P_i^{(k)}) \in \mathcal{D}$. We must show that $R = L_{\mathbb{E}'}(E_i)$.

By definition of the complex $P_i^{(k)}$ there is a triangle

$$E_i \longrightarrow R \longrightarrow A$$

with A in the subcategory $\langle \mathbb{E}' \rangle \subset \mathcal{D}$. Since the complex $P_i^{(k)}$ is a truncation of a projective resolution of the simple module S_i there is also a triangle

$$R \longrightarrow F_i \longrightarrow B$$

with B in the subcategory generated by the elements $E_j \in \mathbb{E}$ with $\phi(E_j) < m - k$. This implies that $R \in \mathbb{E}'^{\perp}$. The result then follows from Remark 2.2(b). \square

Corollary 7.2. *Suppose $\phi: \mathbb{E} \rightarrow \mathbb{Z}$ is tilting at level m and $E_i \in \mathbb{E}_m$. Then the object*

$$L_{\mathbb{E}_{m-k}} \cdots L_{\mathbb{E}_{m-1}}(E_i)[-k]$$

lies in the extension-closed subcategory of \mathcal{D} generated by objects of the form $E[-j]$ for $j \geq 0$ and $E \in (\mathbb{E}_{m-k}, \dots, \mathbb{E}_m)$. Similarly, the object

$$R_{\mathbb{E}_{m+k}} \cdots R_{\mathbb{E}_{m+1}}(E_i)[k]$$

lies in the extension-closed subcategory of \mathcal{D} generated by objects of the form $E[j]$ with $j \geq 0$ and $E \in (\mathbb{E}_m, \dots, \mathbb{E}_{m+k})$.

Proof. The first statement is obvious from the proof of Proposition 7.1 and the explicit form of the complex (13). The second statement is most easily obtained by considering the opposite category \mathcal{D}^{op} . \square

When $k = 1$ the object $P_i^{(1)}[-1]$ is precisely the complex R_i from Section 1.1. It follows that mutations of exceptional collections defined by height functions induce vertex tilts of the corresponding homomorphism algebras.

Proposition 7.3. *Suppose $\phi: \mathbb{E} \rightarrow \mathbb{Z}$ a height function for $E_i \in \mathbb{E}$. Set $(\mathbb{E}', \phi') = \sigma_0(\mathbb{E}, \phi)$ and assume the collection \mathbb{E}' is strong. Then the homomorphism algebra $A(\mathbb{E}')$ is the left tilt of the algebra $A(\mathbb{E})$ at the vertex i .*

Proof. Let Q and Q' be the quivers underlying $A = A(\mathbb{E})$ and $A' = A(\mathbb{E}')$ respectively. The vertices of these quivers correspond to the elements of the collections \mathbb{E} and \mathbb{E}' . Set $E' = L_{\mathbb{E}_{-1}}(E)[-1]$. Since $\mathbb{E} \setminus \{E\} = \mathbb{E}' \setminus \{E'\}$ there is an obvious choice of bijection $\psi: Q_0 \rightarrow Q'_0$. Consider the diagram of equivalences

$$\begin{array}{ccc} \mathcal{D}(A) & \xrightarrow{\Psi} & \mathcal{D}(A') \\ \Phi_{\mathbb{E}} \downarrow & & \downarrow \Phi_{\mathbb{E}'} \\ \mathcal{D} & \xrightarrow{\text{id}} & \mathcal{D} \end{array}$$

Let $i \in Q_0$ be the vertex corresponding to the object E . Applying Lemma 2.6 we see that if $j \in Q_0$ and $j \neq i$ then $\Psi(P_j) = P'_{\psi(j)}$. On the other hand Proposition 7.1 shows that $\Phi_{\mathbb{E}}(R_i) = E'$ and hence $\Psi(R_i) = P_{\psi(i)}$. \square

7.2. Preservation of geometric helices. Our main result shows that geometric helices are preserved under mutations defined by height functions.

Theorem 7.4. *Suppose \mathbb{H} is a geometric helix in \mathcal{D} and $\phi: \mathbb{H} \rightarrow \mathbb{Z}$ is a levelling that is tilting at level m . Set $(\mathbb{H}', \phi') = \sigma_m(\mathbb{H}, \phi)$. Then \mathbb{H}' is geometric.*

Proof. Adding a constant to the levelling we may as well assume that $m = d$. Take a t-structure $\mathcal{D}^{\leq 0} \subset \mathcal{D}$ in which all objects $E \in \mathbb{H}$ satisfying $\phi(E) \geq 0$ are pure. By Remark 2.7 and the fact that \mathbb{H} is geometric, one can construct such a t-structure by considering the equivalence of Theorem 2.4 corresponding to a thread of \mathbb{H} far enough to the left.

Consider the thread $\mathbb{E} = (\mathbb{E}_1, \dots, \mathbb{E}_d) \subset \mathbb{H}$ and the mutated thread

$$\mathbb{E}' = (\mathbb{E}_1, \dots, \mathbb{E}_{d-2}, L_{\mathbb{E}_{d-1}}(\mathbb{E}_d)[-1], \mathbb{E}_{d-1}) \subset \mathbb{H}'.$$

By Lemma 5.2 one has

$$\mathbb{G} = L_{\mathbb{E}_{d-1}}(\mathbb{E}_d)[-1] = R_{\mathbb{E}_{d-2}} \cdots R_{\mathbb{E}_1}(\mathbb{E}_0)[d-2].$$

The two parts of Corollary 7.2 then show that $\mathbb{G} \subset \mathcal{D}^{\geq 0}$ and $\mathbb{G} \subset \mathcal{D}^{\leq 0}$. Thus \mathbb{G} is pure. By periodicity of \mathbb{H} it follows that

$$\mathrm{Hom}_{\mathcal{D}}^{\leq 0}(E_i, E_j) = 0 \text{ for all } i < j.$$

Take an object $A \in \mathbb{G}$. The first part of Corollary 7.2 implies that A is in the extension-closed subcategory generated by right shifts of objects $E \in \mathbb{H}$ with $\phi(E) \leq d$. Since \mathbb{H} is geometric it follows that

$$(14) \quad \mathrm{Hom}_{\mathcal{D}}^{\geq 0}(A, E_j) = 0 \text{ if } \phi(E_j) > 0.$$

Viewing \mathbb{G} instead as a right mutation, the second part of Corollary 7.2 shows that A is in the extension-closed subcategory generated by left shifts of objects $E \in \mathbb{H}$ with $\phi(E) \geq 0$. This gives

$$(15) \quad \mathrm{Hom}_{\mathcal{D}}^{\geq 0}(E_j, A) = 0 \text{ if } \phi(E_j) < d.$$

Finally put $\kappa = S_{\mathcal{D}}[1-d]$ and take $B \in \kappa^r(\mathbb{G})$ for some $r > 0$. By periodicity of the helix, B is in the extension-closed subcategory of \mathcal{D} generated by right shifts of objects $E \in \mathbb{H}$ with $\phi(E) \leq 0$. It follows that $\mathrm{Hom}_{\mathcal{D}}^{\geq 0}(B, A) = 0$. This concludes the proof that the helix generated by \mathbb{E}' is geometric. \square

7.3. Tilting rolled-up helix algebras. In this section we show that mutations of geometric helices defined by height functions induce vertex tilts of the corresponding rolled-up helix algebras. First we prove

Proposition 7.5. *Let \mathbb{H} be a geometric helix of type (n, d) with $d \geq 2$ and let $B(\mathbb{H})$ be the rolled-up helix algebra. Suppose $\phi: \mathbb{H} \rightarrow \mathbb{Z}$ is a height function for $E_i \in \mathbb{H}$. Then the quiver Q underlying $B(\mathbb{H})$ has no loops at the vertex corresponding to E_i . If $d \geq 3$ then there are no 2-cycles in Q passing through this vertex either.*

Proof. Given an integer $k < i$ we claim that every morphism $E_k \rightarrow E_i$ factors via an element of \mathbb{E}_{-1} . To see this consider the triangle

$$(16) \quad L_{\mathbb{E}_{-1}} E_i[-1] \longrightarrow \bigoplus_{E \in \mathbb{E}_{-1}} \mathrm{Hom}_{\mathcal{D}}^{\bullet}(E, E_i) \otimes E \longrightarrow E_i.$$

The existence of such a triangle follows as in the proof of Theorem 5.3, or from Proposition 7.1 together with the explicit form of the complex (13).

If there is a nonzero morphism $E_k \rightarrow E_i$ which does not factor through an element of \mathbb{E}_{-1} then it gives rise to a nonzero element of $\text{Hom}_{\mathbb{D}}^1(E_k, L_{\mathbb{E}_{-1}}(E_i)[-1])$. Since $\phi(E_k) < 0$ this contradicts equation (15) in the proof of Theorem 7.4. Considering right mutations in a similar way shows that if $k > i$ every morphism $E_i \rightarrow E_k$ factors via an element of \mathbb{E}_1 .

Consider now the quiver underlying $B(\mathbb{H})$. Its vertices can be put in bijection with the elements of the thread (E_{i-n+1}, \dots, E_i) . It cannot have loops at the vertex i corresponding to E_i because this would correspond to an irreducible morphism $E_{i-kn} \rightarrow E_i$. Similarly a 2-cycle at i would correspond to irreducible morphisms $E_{i-kn} \rightarrow E_j \rightarrow E_i$. By periodicity of the helix this would imply that $-kd + 1 = \phi(E_j) = -1$ which is impossible for $d \geq 3$. \square

The analogue of Proposition 7.3 relating mutation to tilting is

Proposition 7.6. *Let Z be a smooth Fano variety of dimension $d - 1$ and suppose \mathbb{H} is a geometric helix in $\mathbb{D}(Z)$ of type (n, d) . Suppose there is a height function $\phi: \mathbb{H} \rightarrow \mathbb{Z}$ for an object $E \in \mathbb{H}$ and write $\sigma_0(\mathbb{H}, \phi) = (\mathbb{H}', \phi')$. Then the algebra $B' = B(\mathbb{H}')$ is the left tilt of the algebra $B = B(\mathbb{H})$ at the vertex corresponding to the object E .*

Proof. Take a thread $\mathbb{E} \subset \mathbb{H}$ containing $\mathbb{E}_0 = \{E\}$ and \mathbb{E}_{-1} . The mutated helix \mathbb{H}' is generated by the mutated thread $\mathbb{E}' = \sigma_0(\mathbb{E})$. Let Q and Q' be the quivers underlying $B = B(\mathbb{H})$ and $B' = B(\mathbb{H}')$ respectively. The vertices of these quivers are in natural bijection with the elements of the collections \mathbb{E} and \mathbb{E}' respectively. Set $E' = L_{\mathbb{E}_{-1}}(E)[-1]$. Since $\mathbb{E} \setminus \{E\} = \mathbb{E}' \setminus \{E'\}$ there is an obvious choice of bijection $\psi: Q_0 \rightarrow Q'_0$. Consider the diagram of equivalences

$$\begin{array}{ccc} \mathbb{D}(B) & \xrightarrow{\Psi} & \mathbb{D}(B') \\ \Phi_{\mathbb{E}} \downarrow & & \downarrow \Phi_{\mathbb{E}'} \\ \mathbb{D}(Y) & \xrightarrow{\text{id}} & \mathbb{D}(Y) \end{array}$$

Let $i \in Q_0$ be the vertex corresponding to the object E . Applying Lemma 3.7 we see that if $j \in Q_0$ and $j \neq i$ then $\Psi(P_j) = P'_{\psi(j)}$. It remains to show that $\Phi_{\mathbb{E}}(R_i) = \pi^*(E')$ and hence that $\Psi(R_i) = P'_{\psi(i)}$.

Let $A = A(\mathbb{E})$ be the homomorphism algebra of the thread \mathbb{E} . There is an embedding of graded algebras $i: A \hookrightarrow B$ sending an element of $\text{Hom}_{\mathbb{D}}(E_i, E_j)$ to the element of B obtained by applying all powers of the functor $S_{\mathbb{D}}[1 - d]$. The functor $-\otimes_A B$ sends the projective A -module $e_i A$ to the projective B -module $e_i B$. The existence of this embedding implies the fact already noted in Section 3 that the quiver Q_B underlying B is obtained from the quiver Q_A underlying A by adding extra arrows. The proof of Proposition 7.5 implies that every arrow in Q_B ending at the vertex i actually comes from an arrow of Q_A . It follows that

$$R_i^B = R_i^A \otimes_A B$$

where R_i^A and R_i^B are the objects defined in Section 1.1 for the quiver algebras A and B respectively.

Finally, there is a commuting diagram of functors

$$\begin{array}{ccc} \mathbb{D}(A) & \xrightarrow{-\otimes_A B} & \mathbb{D}(B) \\ \Phi_{\mathbb{E}} \downarrow & & \downarrow \Phi_{\mathbb{E}} \\ \mathbb{D}(Z) & \xrightarrow{\pi^*} & \mathbb{D}(Y) \end{array}$$

Hence the claim follows from Proposition 7.1 with $k = 1$. \square

Remark 7.7. Take assumptions as in Proposition 7.6. Write

$$(\mathbb{H}^L, \phi^L) = \sigma_0(\mathbb{H}, \phi) \text{ and } (\mathbb{H}^R, \phi^R) = \sigma_0^{-1}(\mathbb{H}, \phi).$$

Then $B(\mathbb{H}^L)$ is the left tilt of $B(\mathbb{H})$ at the vertex corresponding to E , and $B(\mathbb{H}^R)$ is the right tilt of $B(\mathbb{H})$ at the vertex corresponding to E .

We claim that in the case $d = 3$ the helices \mathbb{H}^L and \mathbb{H}^R are the same, up to reindexing, and hence $B(\mathbb{H}^L)$ and $B(\mathbb{H}^R)$ are isomorphic algebras. Indeed, if we take the thread

$$(\mathbb{E}_{-1}, E, \mathbb{E}_1) \subset \mathbb{H}$$

then the corresponding threads in \mathbb{H}^L and \mathbb{H}^R are

$$(L_{\mathbb{E}_{-1}}(E)[-1], \mathbb{E}_{-1}, \mathbb{E}_1) \text{ and } (\mathbb{E}_{-1}, \mathbb{E}_1, R_{\mathbb{E}_1}(E)[1]),$$

and the claim follows from Remark 3.2(b).

8. THE CASE OF DEL PEZZO SURFACES

In this section Z is a del Pezzo surface and $\mathbf{D} = \mathbf{D}(Z)$. Following work of Kuleshov and Orlov much is known about exceptional objects on Z . Here we quote their main results following [11] and use an argument of Herzog to construct height functions for strong exceptional collections in \mathbf{D} . Combined with the results of the last section this gives a proof of Theorem 1.7 from the introduction. We conclude by discussing some examples.

8.1. Exceptional objects on del Pezzo surfaces. Given a torsion-free sheaf E on Z we define the slope

$$\mu(E) = \frac{c_1(E) \cdot (-K_Z)}{r(E)}.$$

If E is a torsion sheaf we set $\mu(E) = +\infty$. We say that a torsion-free sheaf E is μ -stable if for all subsheaves $A \subset E$ one has $\mu(A) < \mu(E)$.

Theorem 8.1 (Kuleshov, Orlov). *Every exceptional object in \mathbf{D} is a shift of a sheaf. Moreover every exceptional sheaf A is either a μ -stable locally-free sheaf or a torsion sheaf of the form $\mathcal{O}_C(d)$ with $C \subset Z$ an irreducible rational curve satisfying $C^2 = -1$ and $d \in \mathbb{Z}$ is an integer.*

Proof. This is a combination of Corollary 4.3.2, Theorem 4.3.3 and Proposition 5.3.3 of [11]. \square

Given an exceptional object $F \in \mathbf{D}$ we write $f(F) \in \mathbb{Z}$ for the unique integer such that $F \in \text{Coh}(Z)[f(F)]$.

Suppose now that F_1 and F_2 are two exceptional objects such that $f(F_1) = f(F_2)$. We say $F_1 < F_2$ if

- (a) $\mu(F_1) < \mu(F_2)$, or
- (b) $F_i = \mathcal{O}_C(d_i)$ and $d_1 < d_2$.

If neither $F_1 \leq F_2$ or $F_2 \leq F_1$ holds we write $F_1 \sim F_2$. This is the case precisely when F_1 and F_2 are distinct stable bundles of the same slope, or torsion-sheaves supported on distinct rational curves.

Theorem 8.2. *For any exceptional pair (F_1, F_2) in \mathbf{D} the complex $\text{Hom}_{\mathbf{D}}^\bullet(F_1, F_2)$ is concentrated in a single degree. More precisely, if $f(F_1) = f(F_2)$ then*

$$\begin{aligned} F_1 < F_2 &\implies \text{Hom}_{\mathbf{D}}^k(F_1, F_2) = 0 \text{ unless } k = 0, \\ F_1 > F_2 &\implies \text{Hom}_{\mathbf{D}}^k(F_1, F_2) = 0 \text{ unless } k = 1, \\ F_1 \sim F_2 &\implies \text{Hom}_{\mathbf{D}}^k(F_1, F_2) = 0 \text{ for all } k. \end{aligned}$$

Proof. It is enough to consider the case when F_1 and F_2 are sheaves. Consider the triangle (5)

$$\mathrm{Hom}_{\mathbf{D}}^{\bullet}(F_1, F_2) \otimes F_1 \xrightarrow{ev} F_2 \longrightarrow L_{F_1}(F_2).$$

By Theorem 8.1 the mutated object $L_{F_1}(F_2)$ is of the form $A[n]$ with A a sheaf and $n \in \mathbb{Z}$. Considering the long exact sequence in cohomology it is easy to see that $n \in \{0, 1\}$ and $\mathrm{Hom}_{\mathbf{D}}^{\bullet}(F_1, F_2)$ is concentrated in degrees 0 and 1. If $n = 1$ then $\mathrm{Hom}_{\mathbf{D}}^{\bullet}(F_1, F_2)$ must be concentrated in degree 0. If $n = 0$ there is a long exact sequence of sheaves

$$0 \longrightarrow \mathrm{Hom}_{\mathbf{D}}(F_1, F_2) \otimes F_1 \rightarrow F_2 \xrightarrow{f} L_{F_1}(F_2) \rightarrow \mathrm{Hom}_{\mathbf{D}}^1(F_1, F_2) \otimes F_1 \rightarrow 0.$$

Suppose $\mathrm{Hom}^k(F_1, F_2)$ is nonzero in degrees $k = 0$ and 1 and let P be the cokernel of the map f . The sequence

$$0 \rightarrow P \longrightarrow L_{F_1}(F_2) \rightarrow \mathrm{Hom}_{\mathbf{D}}^1(F_1, F_2) \otimes F_1 \rightarrow 0$$

gives a nonzero element of $\mathrm{Ext}_{\mathbb{Z}}^1(F_1, P)$. The sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathbf{D}}(F_1, F_2) \otimes F_1 \rightarrow F_2 \longrightarrow P \longrightarrow 0$$

then gives a nonzero element of $\mathrm{Ext}_{\mathbb{Z}}^2(F_1, F_1)$ contradicting the fact that F_1 is exceptional. This proves the first statement.

Consider now the second statement. We know from the above that $\mathrm{Hom}_{\mathbf{D}}^{\bullet}(F_1, F_2)$ is concentrated in degrees 0 and 1. Suppose first that F_1 and F_2 are stable bundles of slopes μ_1 and μ_2 respectively. If $\mu_1 \geq \mu_2$ then it is standard that $\mathrm{Hom}_{\mathbf{D}}(F_1, F_2) = 0$. On the other hand if $\mathrm{Hom}_{\mathbf{D}}^1(F_1, F_2)$ is nonzero we have a sequence

$$0 \longrightarrow F_2 \longrightarrow L_{F_1}(F_2) \longrightarrow \mathrm{Hom}_{\mathbf{D}}^1(F_1, F_2) \otimes F_1 \longrightarrow 0$$

and since $L_{F_1}(F_2)$ is a stable bundle by Theorem 8.1, we can conclude that $\mu_1 > \mu_2$. Combining these statements gives the result.

There are three other cases. Suppose first that F_1 and F_2 are both torsion sheaves, supported on rational curves C_1 and C_2 . The exceptional pair assumption gives $\chi(F_2, F_1) = 0$ and Riemann-Roch then implies that $C_1 \cdot C_2 = 0$. Since C_1 and C_2 are irreducible (-1) -curves it follows that they are disjoint, and hence $F_1 \sim F_2$ and $\mathrm{Hom}_{\mathbf{D}}^{\bullet}(F_1, F_2) = 0$.

Next suppose that F_1 is locally-free and $F_2 = \mathcal{O}_C(d)$. Thus $F_1 < F_2$. By Serre duality and the fact that $K_Z \cdot C < 0$ one has

$$\chi(F_1, F_2) = \chi(F_1^* \otimes \mathcal{O}_C(d)) > \chi(F_1^* \otimes \omega_Z \otimes \mathcal{O}_C(d)) = \chi(F_2, F_1) = 0.$$

Hence $\mathrm{Hom}_{\mathbf{D}}^k(F_1, F_2)$ must be concentrated in degree 0. Finally there is the case when $F_1 = \mathcal{O}_C(d)$ and F_2 is locally-free. Then $F_1 > F_2$. Arguing as for the previous case one concludes that $\chi(F_1, F_2) < 0$ and hence $\mathrm{Hom}_{\mathbf{D}}^k(F_1, F_2)$ must be concentrated in degree 1. \square

8.2. Height functions on del Pezzo surfaces. Suppose $\mathbb{E} = (E_1, \dots, E_n)$ is a full strong exceptional collection in \mathbf{D} . In this section we prove that we can reorder \mathbb{E} so that height functions exist for any object $E \in \mathbb{E}$. The argument is due to Herzog [14]. As usual $\mathbb{F} = (F_n, \dots, F_1)$ is the dual collection to \mathbb{E} .

Lemma 8.3. *We can reorder the exceptional collection \mathbb{E} (and hence also the dual collection \mathbb{F}) so that the following holds. Suppose $i \geq j$. Then $f(F_i) \geq f(F_j)$, and if equality holds then either $F_i > F_j$ or $F_i \sim F_j$.*

Proof. Consider neighbouring elements F_{i+1} and F_i in \mathbb{F} . If F_{i+1} and F_i are orthogonal then so are the corresponding objects E_i and E_{i+1} and we can exchange them in the exceptional collection if necessary. So let us assume that this is not the case. Write $F_{i+1} = A[m]$ and $F_i = B[n]$ with A and B sheaves. By Theorem 8.2

$$\mathrm{Hom}_{\mathbb{D}}(A, B) \neq 0 \text{ or } \mathrm{Hom}_{\mathbb{D}}^1(A, B) \neq 0$$

and so $\mathrm{Hom}_{\mathbb{D}}^k(F_{i+1}, F_i)$ is non-vanishing in degree $m - n$ or $m - n + 1$. Since the objects of \mathbb{F} correspond to simple modules under the equivalence $\Phi_{\mathbb{E}}$ one has

$$\mathrm{Hom}_{\mathbb{D}}^k(F_i, F_j) = 0 \text{ for } k < 1$$

so it follows that $m - n \geq 0$.

For the second statement, note that the only other possibility is that $F_i < F_j$. But by Theorem 8.2 and the assumption that F_{i+1} and F_i are not orthogonal this implies that $\mathrm{Hom}_{\mathbb{D}}(F_i, F_j) \neq 0$ which is impossible as before. \square

Let us reorder our collection \mathbb{E} as in Lemma 8.3 and fix an object $E \in \mathbb{E}$. Let $F \in \mathbb{F}$ be the dual object and write $F = A[p]$ with A a sheaf. Split each subcollection $f^{-1}(q) \subset \mathbb{F}$ into two subcollections $f^{-1}(q) = R_q \sqcup L_q$ in such a way that for all $F_j \in f^{-1}(q)$

$$F_j[-q] > A \implies F_j \in R_q \quad F_j[-q] < A \implies F_j \in L_q.$$

We could take $F \in L_p$ or $F \in R_p$, but for definiteness we choose the first possibility. We now have a decomposition of \mathbb{F} of the form

$$\mathbb{F} = (\cdots, R_q, L_q, R_{q-1}, L_{q-1}, \cdots).$$

Of course this induces a decomposition of \mathbb{E} indexed in the opposite direction

$$\mathbb{E} = (\cdots, L_{q-1}, R_{q-1}, L_q, R_q, \cdots).$$

There is thus a levelling $\phi: \mathbb{E} \rightarrow \mathbb{Z}$ defined by

$$\mathbb{E}_q = R_{q-1} \sqcup L_q.$$

It satisfies $\phi(E) = p$. We can now prove

Lemma 8.4. *The levelling $\phi: \mathbb{E} \rightarrow \mathbb{Z}$ is tilting at level p .*

Proof. Recall that $E = A[p]$ with A a sheaf and that $\phi(E) = n$. Take an object $E' \in \mathbb{E}$ with $\phi(E') = p'$ and let $F' \in \mathbb{F}$ be the corresponding dual object. Without loss of generality we can assume that E' comes after E in \mathbb{E} and hence that $p' \geq p$.

Set $m = f(F')$ so that $F' = A'[m]$ for some sheaf A' . Now either $F' \in L_m$ in which case $p' = m$ or $F_j \in R_m$ in which case $p' = m + 1$. In the first case one has $A' \leq A$ or $A' \sim A$. Either way $\mathrm{Hom}_{\mathbb{D}}^{\bullet}(A', A)$ is concentrated in degree zero, so

$$\mathrm{Hom}_{\mathbb{D}}^k(F', F) = 0 \text{ unless } k = p' - p.$$

Similarly, if $F_j \in R_m$ then $B \geq A$ or $B \sim A$ and either way $\mathrm{Hom}_{\mathbb{D}}^{\bullet}(A', A)$ is concentrated in degree 1, so again

$$\mathrm{Hom}_{\mathbb{D}}^k(F', F) = 0 \text{ unless } k = p' - p.$$

Thus E' is $p' - p$ related to E . \square

8.3. Rolled up helix algebras for del Pezzo surfaces. Putting everything together we have now proved

Theorem 8.5. *Let \mathbb{H} be a geometric helix on a del Pezzo surface. Then the rolled-up helix algebra $B = B(\mathbb{H})$ is a graded CY_3 quiver algebra which is noetherian and finite over its centre. The underlying quiver of B has no loops or 2-cycles. For any vertex i of Q there is another geometric helix \mathbb{H}' on Z such that the algebra $B(\mathbb{H}')$ is the (left or right) tilt of $B(\mathbb{H})$ at the vertex i .*

Proof. Suppose \mathbb{H} is a geometric helix on Z and $E \in \mathbb{H}$. By Lemma 6.10 and the existence of height functions for exceptional collections proved in the last section, we know that we can reorder \mathbb{H} so that a height function ϕ exists for $E \in \mathbb{H}$. Of course reordering does not affect the underlying rolled-up helix algebra. The claimed properties of the algebra $B(\mathbb{H})$ then follow from Theorem 3.6 and Proposition 7.5. The statement about tilting follows from Proposition 7.6 and Remark 7.7. \square

As explained in the introduction, the quivers arising via the tilting process can now be completely understood by the cluster mutation rule. It remains to give some examples of geometric helices on del Pezzo surfaces. Recall that any such surface is either $\mathbb{P}^1 \times \mathbb{P}^1$ or the blow-up of \mathbb{P}^2 at $0 \leq m \leq 8$ points. We have already given such examples in the case $Z = \mathbb{P}^2$ and $Z = \mathbb{P}^1 \times \mathbb{P}^1$.

Example 8.6. (a) On the del Pezzo Z which is \mathbb{P}^2 blown up at one point the exceptional collection

$$(\mathcal{O}, \mathcal{O}(h - e), \mathcal{O}(h), \mathcal{O}(2h - e))$$

generates a geometric helix of type $(4, 3)$. Here h is the strict transform of a line in \mathbb{P}^2 and e is the exceptional divisor. The canonical bundle is $\mathcal{O}(-3h + e)$.

(b) On the del Pezzo Z which is \mathbb{P}^2 blown up at two points the exceptional collection

$$(\mathcal{O}, \mathcal{O}(h - e_1), \mathcal{O}(h - e_2), \mathcal{O}(h), \mathcal{O}(2h - e_1 - e_2))$$

generates a geometric helix of type $(5, 3)$. Here again h is the strict transform of a line in \mathbb{P}^2 , and e_1 and e_2 are the exceptional divisors. The canonical bundle is $\mathcal{O}(-3h + e_1 + e_2)$.

(c) On a del Pezzo surface Z which is the blow up of \mathbb{P}^2 in $3 \leq m \leq 8$ points, Karpov and Nogin [17, Proposition 4.2] constructed 3-block exceptional collections of sheaves on Z . By Proposition 5.4 these generate geometric helices of type $(m + 3, 3)$.

APPENDIX A. QUIVER ALGEBRAS AND TILTING

In this section we give sketch proofs of some simple and well-known results about quivers for which we could find no suitable reference. We take notation as in the introduction. In particular a quiver algebra is one of the form

$$A = A(Q, I) = \mathbb{C}Q/I$$

with $I \subset \mathbb{C}Q_{\geq 2}$, and the augmentation ideal $A_+ \subset A$ is spanned by paths of length ≥ 1 .

We use the convention that paths compose on the left, i.e. that $a_1 \cdot a_2 = 0$ unless the target of the arrow a_2 is the source of the arrow a_1 . Thus the space of paths (modulo relations) from vertex i to vertex j is $e_j A e_i$. Since we are considering right-modules this means that a module M determines vector spaces $V_i = M e_i$ for each vertex $i \in Q_0$ and linear maps $V_j \rightarrow V_i$ for each arrow from vertex i to vertex j .

The question of which augmented algebras can be presented as quiver algebras seems to be a tricky one. All we shall need is the following simple result.

Lemma A.1. *Suppose $A = \bigoplus_{n \geq 0} A_n$ is a finitely-generated graded algebra such that $A_0 = \mathbb{S}$ is a finite-dimensional semisimple algebra. Let A_+ denote the augmentation ideal $\bigoplus_{n > 0} A_n$. Then as an augmented algebra (A, A_+) is isomorphic to a quiver algebra $A(Q, I)$. Moreover the quiver Q is uniquely determined by the pair (A, A_+) .*

Proof. Each graded piece A_k is a finite-dimensional \mathbb{S} -bimodule via left and right multiplication by elements of $A_0 = \mathbb{S}$. For each $k \geq 1$ let V_k be the cokernel of the map of \mathbb{S} -bimodules

$$\bigoplus_{0 < j < k} A_j \cdot A_{k-j} \longrightarrow A_k$$

and choose a splitting $i_k: V_k \rightarrow A_k$. Since A is finitely generated we must have $V_k = 0$ for $k \gg 0$ and so $V = \bigoplus_k V_k$ is a finite-dimensional \mathbb{S} -bimodule and $i = \bigoplus i_k$ is an injective map of bimodules $i: V \rightarrow A$. This induces a map of \mathbb{S} -algebras

$$f: T_S(V) \rightarrow A$$

where $T_S(V)$ is the tensor algebra over \mathbb{S} of the bimodule V . Note that $T_S(V)$ is an augmented algebra with augmentation ideal spanned by tensors of positive degree. By construction f is surjective, and has kernel contained in the square of the augmentation.

Choose a basis of orthogonal idempotents (e_1, \dots, e_n) in \mathbb{S} and let Q be the quiver with vertices $\{1, \dots, n\}$ and $\dim_{\mathbb{C}} e_i V e_j$ arrows from vertex i to vertex j . Then it is easy to see that as an augmented algebra $T_S(V)$ is isomorphic to the path algebra $\mathbb{C}Q$. It follows that A is a quiver algebra. Uniqueness follows from equation (1) from the introduction. \square

We shall need the existence of minimal projective resolutions as in the following Lemma.

Lemma A.2. *Suppose $A = A(Q, I)$ is a quiver algebra. For each vertex $i \in Q_0$ let S_i be the corresponding simple module. Then there is a projective resolution of the form*

$$\dots \longrightarrow \bigoplus_{j \in Q_0} P_j^{\oplus d_{j,i}^k} \longrightarrow \dots \longrightarrow \bigoplus_{j \in Q_0} P_j^{\oplus d_{j,i}^1} \longrightarrow P_i \longrightarrow S_i \longrightarrow 0$$

where $d_{j,i}^p = \dim_{\mathbb{C}} \text{Ext}_A^p(S_i, S_j)$.

Proof. An alternative way to state this is that one can construct a projective resolution such that when one applies the functor $\text{Hom}_A(-, S_j)$ all maps become zero. One can build such a resolution step-by-step. All one needs to know is that for any finitely-generated module M there is a projective module

$$P = \bigoplus_{k \in Q_0} P_k^{n_k}$$

and a surjection $f: P \rightarrow M$ such that for each j the induced map

$$f^*: \text{Hom}_A(M, S_j) \rightarrow \text{Hom}_A(P, S_j)$$

is a surjection. To prove this write $M = \bigoplus_{i \in Q_0} M_i$ where $M_i = M e_i$ and take elements $m_i \in M_i$ such that the images in $M/A_+ M$ form a basis. Each element m_i defines a map $P_i \rightarrow M$, and the corresponding map $\bigoplus P_i \rightarrow M$ has the required property. \square

Remark A.3. If Q has no oriented cycles then it is possible to order the vertices of the quiver so that $\text{Hom}_A(P_i, P_j) = 0$ unless $i < j$. It follows that the projective resolutions above must be finite, and hence A has finite global dimension.

Finally we prove the claim made in Remark 1.2. Suppose $A = A(Q, I)$ and $A' = A(Q', I')$ are quiver algebras related by a tilt at the vertex i as in Definition 1.1. Assume that their underlying quivers have no loops.

Lemma A.4. *Define objects $U_j \in D_{\text{fin}}(A)$ by the relation $\Psi(U_j) = S_{\psi(j)}$. Then $U_i = S_i[-1]$, whereas for $j \neq i$ the object U_j is the universal extension*

$$(17) \quad 0 \longrightarrow S_j \longrightarrow U_j \longrightarrow \text{Ext}_A^1(S_i, S_j) \otimes S_i \longrightarrow 0.$$

Proof. Let U_j be the given objects; we will prove that $\Psi(U_j) = S_{\psi(j)}$. Define an object

$$(18) \quad T = R_i \oplus \bigoplus_{i \neq j \in Q_0} P_j \in D(A).$$

Then $\Psi(T) = A'$ and to prove the claim we must check that $\text{Hom}_A(T, U_j) = \mathbb{C}$ for each vertex $j \in Q_0$. The only tricky thing is to show that $\text{Hom}_A(R_i, U_j) = 0$ for $j \neq i$. Let V_{ij} be the n_{ij} -dimensional vector space spanned by the arrows from i to j . Then since we are considering right-modules $\text{Ext}_A^1(S_i, S_j) = V_{ji}^*$. We must show that the canonical map $P_j \otimes V_{ji} \rightarrow P_i$ induces an isomorphism

$$(19) \quad V_{ji}^* = \text{Hom}_A(P_i, U_j) \longrightarrow \text{Hom}_A(P_j \otimes V_{ji}, U_j) = V_{ji}^*.$$

Viewed as a representation of the quiver Q , the object U_j can be represented by associating the vector space V_{ji}^* to the vertex i , the one-dimensional vector space \mathbb{C} to the vertex j , and the tautological linear map $V_{ji}^* \rightarrow \mathbb{C}$ to each arrow $a \in V_{ji}$. On the other hand $P_i = e_i A$ associates to each vertex j the space of paths in Q (modulo relations) from j to i . For each element $\xi \in V_{ji}^*$ there is a map $P_i \rightarrow U_j$ defined by sending the lazy path at i to $\xi \in V_{ji}^*$, sending an arrow $a \in V_{ji}$ to $\xi(a) \in \mathbb{C}$, and sending all other paths to zero. Composing with the canonical map $P_j \otimes V_{ji} \rightarrow P_i$ gives a nonzero map for all nonzero ξ . This shows that (19) is injective, and hence an isomorphism. \square

APPENDIX B. SEMI-ORTHOGONAL DECOMPOSITION

Here we recall Bondal's categorical approach to mutation functors. For more details we refer the reader to [4]. Throughout D denotes an arbitrary \mathbb{C} -linear triangulated category of finite type.

Suppose $\mathcal{A} \subset D$ is a full subcategory. The *right orthogonal subcategory* to \mathcal{A} is

$$\mathcal{A}^\perp = \{X \in D : \text{Hom}_D^\bullet(A, X) = 0 \text{ for } A \in \mathcal{A}\} \subset D.$$

Similarly, the *left orthogonal subcategory* to \mathcal{A} is

$${}^\perp \mathcal{A} = \{X \in D : \text{Hom}_D^\bullet(X, A) = 0 \text{ for } A \in \mathcal{A}\} \subset D.$$

Both are full triangulated subcategories of D .

A *semi-orthogonal decomposition* of D is a pair of full triangulated subcategories $(\mathcal{A}, \mathcal{B}) \subset D$ such that

- (a) for $A \in \mathcal{A}$ and $B \in \mathcal{B}$ one has $\text{Hom}_D(B, A) = 0$,
- (b) for every object $X \in D$ there is triangle

$$B \longrightarrow X \longrightarrow A$$

such that $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

A full triangulated subcategory $\mathcal{A} \subset D$ is *left or right admissible* if the inclusion functor $\mathcal{A} \hookrightarrow D$ has a left or right adjoint respectively.

Proposition B.1 (Bondal). *Suppose $\mathcal{A}, \mathcal{B} \subset D$ are full, triangulated subcategories closed under isomorphism. Then the following are equivalent*

- (a) $(\mathcal{A}, \mathcal{B}) \subset D$ is a semi-orthogonal decomposition,

- (b) \mathcal{A} is left admissible and $\mathcal{B} = {}^\perp \mathcal{A}$,
- (c) \mathcal{B} is right admissible and $\mathcal{A} = \mathcal{B}^\perp$.

Proof. This can be found in [4]. Here we just sketch the argument. First assume that $(\mathcal{A}, \mathcal{B})$ is a semi-orthogonal decomposition. Condition (a) in the definition of semi-orthogonality implies that for any $X \in \mathcal{D}$ the triangle appearing in part (b) is unique up to a unique isomorphism. This implies that there are functors

$$p: \mathcal{D} \rightarrow \mathcal{A}, \quad q: \mathcal{D} \rightarrow \mathcal{B},$$

sending an object $X \in \mathcal{D}$ to the objects $A \in \mathcal{A}$ and $B \in \mathcal{B}$ respectively. It is then easy to see that p is the left adjoint to the inclusion $\mathcal{A} \hookrightarrow \mathcal{D}$ and that similarly q is the right adjoint to the inclusion $\mathcal{B} \hookrightarrow \mathcal{D}$.

Finally, note that if $X \in {}^\perp \mathcal{A}$ then the map $X \rightarrow A$ must be zero. Thus $\text{id}_A: A \rightarrow A$ factors via $B[1] \in \mathcal{B}$ and hence is zero by the orthogonality condition. This implies that ${}^\perp \mathcal{A} \subset \mathcal{B}$. The opposite inclusion is immediate from part (a) of the definition. Similarly $\mathcal{B}^\perp = \mathcal{A}$, so (a) implies (b) and (c).

For the converse we must prove that (a) is implied by either (b) or (c). Without loss of generality assume (b) so that the inclusion $i: \mathcal{A} \rightarrow \mathcal{D}$ has a left adjoint $p: \mathcal{D} \rightarrow \mathcal{A}$. Taking the cone on the unit of the adjunction gives for any object $X \in \mathcal{D}$ a triangle

$$X \longrightarrow (i \circ p)(X) \longrightarrow B.$$

Since i is fully faithful $p \circ i \cong \text{id}_{\mathcal{A}}$ and so applying p to the above triangle shows that $p(B) = 0$. Now

$$\text{Hom}_{\mathcal{D}}(B, i(A)) = \text{Hom}_{\mathcal{A}}(p(B), A) = 0$$

so $B \in {}^\perp \mathcal{A} = \mathcal{B}$. This proves that $(\mathcal{A}, \mathcal{B})$ is a semi-orthogonal decomposition. \square

A full triangulated subcategory $\mathcal{A} \subset \mathcal{D}$ is called *admissible* if it both left and right admissible. Then by Proposition B.1 one has semi-orthogonal decompositions $(\mathcal{A}, {}^\perp \mathcal{A})$ and $(\mathcal{A}^\perp, \mathcal{A})$. Let $p: \mathcal{D} \rightarrow {}^\perp \mathcal{A}$ be the left adjoint to the inclusion functor $i: {}^\perp \mathcal{A} \rightarrow \mathcal{D}$, and let $q: \mathcal{D} \rightarrow \mathcal{A}^\perp$ be the right adjoint to the inclusion $j: \mathcal{A}^\perp \rightarrow \mathcal{D}$.

$$\begin{array}{ccccc} {}^\perp \mathcal{A} & \xrightarrow{i} & \mathcal{D} & \xrightarrow{q} & \mathcal{A}^\perp \\ & \searrow p & & \swarrow j & \\ & & & & \end{array}$$

The composite functors $L_{\mathcal{A}} = q \circ i$ and $R_{\mathcal{A}} = p \circ j$ are called the *mutation functors* for the subcategory $\mathcal{A} \subset \mathcal{D}$.

Lemma B.2. *Suppose $X \in {}^\perp \mathcal{A}$ and $Y \in \mathcal{A}^\perp$. Then $Y = L_{\mathcal{A}}(X)$ iff there is a triangle*

$$A \longrightarrow X \longrightarrow Y$$

with $A \in \mathcal{A}$. Similarly $X = R_{\mathcal{A}}(Y)$ iff there is a triangle

$$X \longrightarrow Y \longrightarrow A'$$

with $A' \in \mathcal{A}$.

Proof. This follows immediately from the proof of Proposition B.1. \square

Rotating the triangle it is then obvious that the mutation functors $L_{\mathcal{A}}$ and $R_{\mathcal{A}}$ are mutually-inverse equivalences of categories.

A saturated triangulated category is always admissible in any enveloping category. In particular, if $\mathbb{E} \subset \mathcal{D}$ is an exceptional collection then $\langle \mathbb{E} \rangle \subset \mathcal{D}$ is always admissible. It follows immediately from Lemma B.2 that if $\mathbb{E} \subset \mathcal{D}$ is an exceptional collection, then the mutation functors $L_{\mathbb{E}}$ and $R_{\mathbb{E}}$ defined in Section 2 coincide with the mutation functors $L_{\langle \mathbb{E} \rangle}$ and $R_{\langle \mathbb{E} \rangle}$ defined above.

APPENDIX C. EXCEPTIONAL COLLECTIONS

Here we give the proofs of various simple results on exceptional collections from Section 2. Assumptions are as in Section 2.1.

Lemma C.1. *Let $\mathbb{E} \subset \mathcal{D}$ be an exceptional collection. Then the following are equivalent*

- (a) $\langle \mathbb{E} \rangle = \mathcal{D}$,
- (b) $\mathbb{E}^{\perp} = 0$,
- (c) ${}^{\perp}\mathbb{E} = 0$.

Proof. Take $X \in \mathbb{E}^{\perp}$. Then $\mathrm{Hom}_{\mathcal{D}}^{\bullet}(E, X) = 0$ for all $E \in \mathbb{E}$ and so $\mathrm{Hom}_{\mathcal{D}}^{\bullet}(Y, X) = 0$ for all $Y \in \langle \mathbb{E} \rangle$. If (a) holds we can take $Y = X$ and so $X = 0$. Thus (a) implies (b) and similarly (c). For the converse note that $\langle \mathbb{E} \rangle$ is saturated and hence is admissible in \mathcal{D} . Thus there is a semi-orthogonal decomposition $(\mathbb{E}^{\perp}, \langle \mathbb{E} \rangle)$. If (b) holds then it follows that $\langle \mathbb{E} \rangle = \mathcal{D}$. \square

Lemma C.2. *Let $\mathbb{E} = (E_1, \dots, E_n)$ be a full exceptional collection in \mathcal{D} . For any $1 \leq k \leq n$ define exceptional collections $\mathbb{E}_{\leq k} = (E_1, \dots, E_k)$ and $\mathbb{E}_{> k} = (E_{k+1}, \dots, E_n)$. Then for each k there is a semi-orthogonal decomposition*

$$\langle \mathbb{E} \rangle = (\langle \mathbb{E}_{\leq k} \rangle, \langle \mathbb{E}_{> k} \rangle).$$

Proof. The subcategory $\langle \mathbb{E}_{> k} \rangle$ is saturated and hence admissible. There is thus a semi-orthogonal decomposition $(\mathbb{E}_{> k}^{\perp}, \langle \mathbb{E}_{> k} \rangle)$. Clearly $\mathbb{E}_{\leq k}$ is an exceptional collection in $\mathbb{E}_{> k}^{\perp}$ and since \mathbb{E} is full in \mathcal{D} it follows from Lemma C.1 that $\mathbb{E}_{\leq k}$ is full in $\mathbb{E}_{> k}^{\perp}$. This gives the result. \square

Lemma C.3. *If \mathcal{D} contains a full exceptional collection of length n then $K(\mathcal{D}) \cong \mathbb{Z}^{\oplus n}$.*

Proof. If $E \in \mathcal{D}$ is an exceptional object then there is an equivalence $\mathcal{D}(\mathbb{C}) \rightarrow \langle E \rangle$ sending \mathbb{C} to E . In particular $K(\langle E \rangle) = \mathbb{Z}$ is generated by the class $[E]$. This can be seen as a special case of Theorem 2.4 but is also easy to check directly. Now suppose $\mathbb{E} = (E_1, \dots, E_n)$ is a full exceptional collection in \mathcal{D} . Applying Lemma C.2 repeatedly it follows that $K(\mathcal{D})$ is spanned by the classes $[E_i]$. Recall that the Euler form

$$\chi(X, Y) = \sum_{i \in \mathbb{Z}} (-1)^i \dim_{\mathbb{C}} \mathrm{Hom}_{\mathcal{D}}^i(X, Y)$$

descends to $K(\mathcal{D})$. Since $\chi(E_i, E_j) = 0$ for $i > j$ and $\chi(E_i, E_i) = 1$ it follows that the classes $[E_i] \in K(\mathcal{D})$ are linearly independent in $K(\mathcal{D})$ and hence form a basis. \square

Proof of Lemma 2.9. Let $q, r: \mathcal{D} \rightarrow \mathbb{E}^{\perp}$ be the left and right adjoints to the inclusion functor $j: \mathbb{E}^{\perp} \rightarrow \mathcal{D}$ respectively. The first claim is that

$$S_{\mathbb{E}^{\perp}} \circ q \cong r \circ S_{\mathcal{D}}.$$

This follows from the definition of the Serre functor. Indeed, writing $\mathcal{C} = \mathbb{E}^\perp$ one has

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}}(X, S_{\mathcal{C}}(q(Y))) &= \mathrm{Hom}_{\mathcal{C}}(q(Y), X)^* = \mathrm{Hom}_{\mathbb{D}}(Y, j(X))^* \\ &= \mathrm{Hom}_{\mathbb{D}}(j(X), S_{\mathbb{D}}(Y)) = \mathrm{Hom}_{\mathcal{C}}(X, r(S_{\mathbb{D}}(Y))) \end{aligned}$$

for $X \in \mathcal{C}$ and $Y \in \mathbb{D}$. Let $i: {}^\perp\mathbb{E} \rightarrow \mathbb{D}$ be the inclusion functor. Using the description $L_{\mathbb{E}} = q \circ i$ from the last section, the equivalence (9) can be rewritten as

$$S_{\mathbb{D}}|_{{}^\perp\mathbb{E}} \cong p \circ S_{\mathbb{D}} \circ i \cong S_{{}^\perp\mathbb{E}} \circ q \circ i \cong S_{{}^\perp\mathbb{E}} \circ L_{\mathbb{E}}.$$

This gives the result. \square

Proof of Lemma 2.5. Write $\mathbb{E}_{\leq j}$ for the subcollection $(E_1, \dots, E_j) \subset \mathbb{E}$ and put $\mathbb{E}_{< j} = \mathbb{E}_{\leq j-1}$. Then $F_j = L_{\mathbb{E}_{< j}}(E_j)$ and is therefore an object of the subcategory

$$\mathbb{E}_{< j}^\perp \subset \langle \mathbb{E}_{\leq j} \rangle.$$

It follows immediately that \mathbb{F} is an exceptional collection, and that

$$\mathrm{Hom}_{\mathbb{D}}^k(E_i, F_j) = 0 \text{ for } i \neq j.$$

It is also easy to see that the collection \mathbb{F} is full. The triangle of Lemma B.2 takes the form

$$Y \longrightarrow E_j \longrightarrow F_j$$

for some $Y \in \mathbb{E}_{< j}$. Since $E_i \in {}^\perp\mathbb{E}_{< i}$, applying the functor $\mathrm{Hom}_{\mathbb{D}}(E_i, -)$ shows that

$$\mathrm{Hom}_{\mathbb{D}}^\bullet(E_i, F_i) = \mathbb{C}.$$

This completes the proof of the first part of the Lemma.

For the uniqueness statement note first that by assumption $F_j \in \mathbb{E}_{< j}^\perp$. Let Y be the cone on a nonzero map $E_j \rightarrow F_j$ fitting into a triangle

$$Y \longrightarrow E_j \longrightarrow F_j$$

as before. Applying the functor $\mathrm{Hom}_{\mathbb{D}}(E_j, -)$ shows that $\mathrm{Hom}(E_j, Y) = 0$ and therefore by Lemma C.2

$$Y \in \mathbb{E}_{\geq j}^\perp = \langle \mathbb{E}_{< j} \rangle.$$

Applying Lemma B.2 it follows that $F_j = L_{\mathbb{E}_{< j}}(E_j)$.

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